We get the roots by

```
EDU>> roots(pp)
ans =
    10.0606
     4.0644
    -1.1250
```

which is the same as before, as expected.

Homework 1: Solve problems 1, 29, 46, APP11 Deadline: 28 Apr 2019

Exercises

Section 6.1

1. Use the Taylor series method to get solutions to

$$dy/dx = x + y - xy, \qquad y(0) = 1$$

at $x = 0.1$ and $x = 0.5$. Use terms through x^5.

▶ 2. The solution to Exercise 1 at $x = 0.5$ is 1.59420. How many terms of a Taylor series must be used to match this?

3. Repeat Exercises 1 and 2 but for

$$y''(x) = x/y, \qquad y(0) = 1, \qquad y'(0) = 1.$$

The correct value for $y(0.5)$ is 1.51676.

4. A spring system has resistance to motion proportional to the square of the velocity, and its motion is described by

$$\frac{d^2x}{dt^2} + 0.1\left(\frac{dx}{dt}\right)^2 + 0.6x = 0.$$

If the spring is released from a point that is a unit distance above its equilibrium point, $x(0) = 1$, $x'(0) = 0$, use the Taylor-series method to write a series expression for the displacement as a function of time, including terms up to t^6.

Section 6.2

5. Repeat Exercise 1, but use the simple Euler method. How small must h be to match to the values of Exercise 1?

▶ 6. Repeat Exercise 2, but use the simple Euler method. How small must h be?

7. Repeat Exercise 5, but now with the modified Euler method. Comparing to Exercise 5, how much less effort is required?

8. Find the solution to

$$\frac{dy}{dt} = y^2 + t^2, \quad y(1) = 0, \quad \text{at } t = 2,$$

by the modified Euler method, using $h = 0.1$. Repeat with $h = 0.05$. From the two results, estimate the accuracy of the second computation.

9. Solve $y' = \sin(x) + y$, $y(0) = 2$ by the modified Euler method to get $y(0.1)$ and $y(0.5)$. Use a value of h small enough to be sure that you have five digits correct.

▶ 10. A sky diver jumps from a plane, and during the time before the parachute opens, the air resistance is proportional to the $\frac{3}{2}$ power of the diver's velocity. If it is known that the maximum rate of fall under these conditions is 80 mph, determine the diver's velocity during the first 2 sec of fall using the modified Euler method with $\Delta t = 0.2$. Neglect horizontal drift and assume an initial velocity of zero.

11. Repeat Exercise 8 but use the midpoint method. Are the results the same? If not, which is more accurate?

12. The midpoint method gives results identical to modified Euler for $dy/dx = -2x - xy$, $y(0) = -1$. But for some definitions of dy/dx, it is better; for other definitions, it is worse. What are the conditions on the derivative function that cause

a. The midpoint method to be better?
b. The midpoint method to be poorer?
c. The two methods to give identical results?
d. Give specific examples for parts (a) and (b).

▶ 13. For some derivative functions, the simple Euler method will have errors that are always positive but for others, the errors will always be negative.

a. What property of the function will determine which kind of error will be experienced?
b. Provide examples for both types of derivative function.

c. When will the errors be positive at first, but then become negative? Give an example where the errors oscillate between positive and negative as the x-values increase.

14. Is the phenomenon of Exercise 13 true for the modified Euler method? If it is, repeat Exercise 13 for this method.

Section 6.3

15. What are the equations that will be used for a second-order Runge–Kutta method if $a = 1/3$, $b = 2/3$, $\alpha = 3/4$ and $\beta = 3/4$. The statement is made that "this is said to give a minimum bound to the error." Test the truth of this statement by comparing this method with modified Euler on the equations of Exercises 1 and 8. Also compare to the midpoint method.

16. What is the equivalent of Eq. (6.10) for a third-order RK method? What then is the equivalent of Eq. (6.12)? Give three different combinations of parameter values that can be employed.

17. Use one set of the parameter values you found in Exercise 16 to solve Exercise 9.

a. How much larger can h be than the value found in Exercise 9?

b. Repeat with the other sets of parameters. Which set is preferred?

18. Solve Exercise 1 with fourth-order Runge–Kutta method. How large can h be to get the correct value at $x = 1.0$, which is 2.19496?

19. Determine y at $x = 1$ for the following equation, using fourth-order Runge–Kutta method with $h = 0.2$. How accurate are the results?

$$dy/dx = 1/(x + y), \qquad y(0) = 2.$$

▶**20.** Using the conditions of Exercise 10, determine how long it takes for the jumper to reach 90% of his or her maximum velocity, by integrating the equation using the Runge–Kutta technique with $\Delta t = 0.5$ until the velocity exceeds this value, and then interpolating. Then use numerical integration on the velocity values to determine the distance the diver falls in attaining $0.9v_{\text{max}}$.

21. It is not easy to know the accuracy with which the function has been determined by either the Euler methods or the Runge–Kutta method. A possible way to measure accuracy is to repeat the problem with a smaller step size, and compare results. If the two computations agree to n decimal places, one then assumes the values are correct to that many places. Repeat Exercise 20 with $\Delta t = 0.3$, which should give a global error about one-eighth as large, and by comparing results, determine the accuracy in Exercise 20. (Why do we expect to reduce the error eightfold by this change in Δt?)

22. Solve Exercises 1, 9, and 10 by the Runge–Kutta–Fehlberg method.

23. Using Runge–Kutta–Fehlberg, compare your results to that from fourth-order Runge–Kutta method in Exercise 18.

▶**24.** Solve $y' = 2x^2 - y$, $y(0) = -1$ by the Runge–Kutta–Fehlberg method to $x = 2.0$. How large can h be and still get the solution accurate to 6 significant digits?

25. Add the results from the Runge–Kutta–Fehlberg method to Table 6.6.

26. In the algorithm for the Runge–Kutta–Fehlberg method, an expression for the error is given. Repeat Exercise 19 with the Runge–Kutta–Fehlberg method and compare the actual error to the value from the expression.

Section 6.4

▶**27.** Derive the formula for the second-order Adams method. Use the method of undetermined coefficients.

28. Use the formula of Exercise 27 to get values as in Example 6.1.

29. For the differential equation

$$\frac{dy}{dt} = y - t^2, \quad y(0) = 1,$$

starting values are known:

$$y(0.2) = 1.2186, \qquad y(0.4) = 1,4682,$$
$$y(0.6) = 1.7379.$$

Use the Adams method, fitting cubics with the last four (y, t) values and advance the solution to $t = 1.2$. Compare to the analytical solution.

▶**30.** For the equation

$$\frac{dy}{dt} = t^2 - t, \quad y(1) = 0,$$

the analytical solution is easy to find:

$$y = \frac{t^3}{3} - \frac{t^2}{2} + \frac{1}{6}.$$

If we use three points in the Adams method, what error would we expect in the numerical solution? Confirm your expectation by performing the computations.

31. Repeat Exercise 30, but use four points.

32. Solve Exercise 29 with Adams–Moulton fourth order method.

33. For the equation $y' = y * \sin(\pi x)$, $y(0) = 1$, get starting values by RKF for $x = 0.2$, 0.4, and 0.6 and then advance the solution to $x = 1.4$ by Adams–Moulton fourth order method.

34. Get the equivalent of Eqs. (6.16) and (6.17) for a third-order Adams–Moulton method.

35. Derive the interpolation formulas given in Section 6.4 that permit getting additional values to reduce the step size.

▶36. Use Eq. (6.18) on this problem

$$dy/dx = 2x + 2, \quad y(1) = 3.$$

a. Is instability indicated?

b. Compare the results with this method to those from the simple Euler method as in Tables 6.11 and 6.12.

37. Use Milne's method on the equation in Exercise in 36. Is there any indication of instability?

38. Parallel the theoretical demonstration of instability with Milne's method with the equation $dy/dx = Ax^n$, where A and n are constants. What do you conclude?

39. What is the error term for Hamming's method? Show that it is a stable method.

Section 6.5

40. The mathematical model of an electrical circuit is given by the equation

$$0.5\frac{d^2Q}{dt^2} + 6\frac{dQ}{dt} + 50Q = 24 \sin 10t,$$

with $Q = 0$ and $i = dQ/dt = 0$ at $t = 0$. Express as a pair of first-order equations.

▶41. In the theory of beams, it is shown that the radius of curvature at any point is proportional to the bending moment:

$$EI\frac{y''}{[1 + (y')^2]^{3/2}} = M(x),$$

where y is the deflection of the neutral axis. In the usual approach, $(y')^2$ is neglected in comparison to unity, but if the beam has appreciable curvature, this is invalid. For the cantilever beam for which $y(0) = y'(0) = 0$, express the equation as a pair of simultaneous first-order equations.

42. A cantilever beam is 12 ft long and bears a uniform load of W lb/in. so that $M(x) = W * x^2/2$. Exercise 41 suggests that a simplified version of the differential equation can be used if the curvature of the beam is small. For what value of W, the value of the uniform load, does the simplified equation give a value for the deflection at the end of the beam that is in error by 5%?

▶43. Solve the pair of simultaneous equations

$$dx/dt = xy - t, \quad x(0) = 1,$$
$$dy/dt = x + t, \quad y(0) = 0,$$

by the modified Euler method from $t = 0$ to $t = 1.0$ in steps of 0.2.

44. Repeat Exercise 43, but with the Runge–Kutta–Fehlberg method. How accurate are these results? How much are the errors less than those of Exercise 43?

45. Use the first results of Exercise 44 to begin the Adams–Moulton method and then advance the solution to $x = 1.0$. Are the results as accurate as with the Runge–Kutta–Fehlberg method?

▶46. The motion of the compound spring system as sketched in Figure 6.7 is given by the solution of the pair of simultaneous equations

$$m_1\frac{d^2y_1}{dt^2} = -k_1y_1 - k_2(y_1 - y_2),$$
$$m_2\frac{d^2y_2}{dt^2} = k_2(y_1 - y_2),$$

where y_1 and y_2 are the displacements of the two masses from their equilibrium positions. The initial conditions are

$$y_1(0) = A, \quad y_1'(0) = B, \quad y_2(0) = C, \quad y_2'(0) = D.$$

Express as a set of first-order equations.

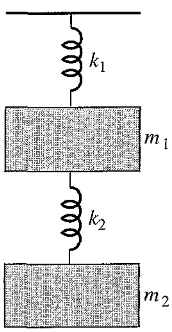

Figure 6.7

47. For the third-order equation

$$y''' + ty' - 2y = t, \quad y(0) = y''(0) = 0, \quad y'(0) = 1,$$

a. Solve for $y(0.2)$, $y(0.4)$, $y(0.6)$ by RKF.
b. Advance the solution to $t = 1.0$ with the Adams–Moulton method.
c. Estimate the accuracy of $y(1.0)$ in part (b).

48. Solve the equation in Exercise 47 by the Taylor-series method. How many terms are needed to be sure that $y(1.0)$ is correct to four significant digits?

49. If some simplifying assumptions are made, the equations of motion of a satellite around a central body are

$$\frac{d^2x}{dt^2} = \frac{-x}{r^3}, \quad \frac{d^2y}{dt^2} = \frac{-y}{r^3},$$

where

$$r = \sqrt{(x^2 + y^2)}, \qquad x(0) = 0.4,$$
$$y(0) = x'(0) = 0, \qquad y'(0) = 2.$$

a. Evaluate $x(t)$ and $y(t)$ from $t = 0$ to $t = 10$ in steps of 0.2. Use any of the single-step methods to do this.
b. Plot the curve for this range of t-values.
c. Estimate the period of the orbit.

Section 6.6

50. Equation 6.22 is for a *stiff* equation. If the coefficients of the equation for x' are changed, for what values is the system no longer stiff?

51. A pair of differential equations has the solution

$$x(t) = e^{-22t} - e^{-t},$$
$$y(t) = e^{-22t} + e^{-t},$$

with initial conditions of $x(0) = 0$, $y(0) = 2$.

a. What are the differential equations?
b. Is that system "stiff"?
c. What are the computed values for $x(0.2)$ and $y(0.2)$ if the equations of part (a) are solved with the simple Euler method, with $h = 0.1$?
d. Repeat part (c), but employing the method of Eq. (6.23). Is this answer closer to the correct value?
e. How small must h be to get the solutions at $t = 0.2$ accurate to four significant digits when using the simple Euler method?
f. Repeat part (e), but now for the method of Eq. (6.23).

▶**52.** When testing a linear system to see if it is "stiff" it is convenient to write it as

$$\begin{bmatrix} x \\ y \end{bmatrix}' = A \begin{bmatrix} x \\ y \end{bmatrix},$$

where the elements of matrix A are the multipliers of x and y in the equations. If the eigenvalues of A are all real and negative and differ widely in magnitude, the system is stiff. (One can get the eigenvalues from the characteristic polynomial as explained in Chapter 2 or with a computer algebra system.)

Suppose that A has these elements:

$$A = \begin{bmatrix} 19 & -20 \\ -20 & 19 \end{bmatrix}.$$

a. What are the eigenvalues of A? Would you call the system stiff?
b. Change the elements of A so that all are positive. What are the eigenvalues of A after this change? Does this make the system "nonstiff"?

53. The definition of a stiff equation as one whose coefficient matrix has negative eigenvalues that "differ widely in magnitude" is rather subjective. Propose an alternate definition of stiffness that is more specific.

Section 6.7

54. Suppose that a rod of length L is made from two dissimilar materials welded together end-to-end. From $x = 0$ to $x = X$, the thermal conductivity is k_1; from $x = X$ to $x = L$, it is k_2. How will the temperatures vary along the rod if $u = 0°$ at $x = 0$ and $u = 100°$ at $x = L$? Assume that Eq. (6.24) applies with $Q = 0$ and that the cross-section is constant.

55. What if k varies with temperature: $k = a + bu + cu^2$? What is the equation that must be solved to determine the temperature distribution along a rod of constant cross section?

56. Solve the boundary value problem

$$d^2x/dt^2 + t\,(dx/dt) - 3x = 3t, \quad x(0) = 1, \quad x(2) = 5$$

by "shooting." (The initial slope is near -1.5.) Use $h = 0.25$ and compare the results from the Runge–Kutta–Fehlberg method and modified Euler methods. Why are the results different? Is it possible to match the Runge–Kutta–Fehlberg method results when the modified Euler method is used? If so, show how this can be accomplished.

57. Repeat Exercise 56, but with smaller values for h. At what h-values with the Runge–Kutta–Fehlberg method are successive computations the same?

58. The boundary-value problem of Exercise 56 is linear. That means that the correct initial slope can be found by interpolating from two trial values. Show that intermediate values from the computations obtained with these two trial values can themselves be interpolated to get correct intermediate values for $x(t)$.

59. If the equation of Exercise 56 is changed only slightly to

$$d^2x/dt^2 + x\,(dx/dt) - 3x = 3t, \quad x(0) = 1, \quad x(2) = 5,$$

it is no longer linear. Solve it by the shooting method using RKF. Do you find that more than two trials are needed to get the solution? What is the correct value for the initial slope? Use a value of h small enough to be sure that the results are correct to five significant digits.

60. Given this boundary-value problem:

$$\frac{d^2y}{d\theta^2} + \frac{y}{4} = 0, \quad y(0) = 0, \quad y(\pi) = 2,$$

which has the solution $y = 2\sin(\theta/2)$,

a. Solve, using finite difference approximations to the derivative with $h = \pi/4$ and tabulate the errors.
b. Solve again by finite differences but with a value of h small enough to reduce the maximum error to 0.5%. Can you predict from part (a) how small h should be?
c. Solve again by the shooting method. Find how large h can be to have maximum error of 0.5%.

61. Solve Exercise 56 though a set of equations where the derivatives are replaced by difference quotients. How small must h be to essentially match to the results of Exercise 56 when RKF was used?

62. Use finite difference approximations to the derivatives to solve Exercise 59. The equations will be nonlinear so they are not as easily solved. One way to approach the solution is to linearize the equations by replacing x in the second term with an approximate value, then using the results to refine this approximation successively. Solve it this way.

63. Solve this boundary-value problem by finite differences, first using $h = 0.2$, then with $h = 0.1$:

$$y'' + xy' - x^2y = 2x^3, \quad y(0) = 1, \quad y(1) = -1.$$

Assuming that errors are proportional to h^2, extrapolate to get an improved answer. Then, using a very small h-value in the shooting method, see if this agrees with your improved answer.

64. Repeat Exercise 60, except with these derivative boundary conditions:

$$y'(0) = 0, \qquad y'(\pi) = 1.$$

In part (a), compare to $y = -2\cos(\theta/2)$.

65. Solve through finite differences with four subintervals:

$$\frac{d^2y}{dx^2} + y = 0, \quad y'(0) + y(0) = 2,$$

$$y'\left(\frac{\pi}{2}\right) + y\left(\frac{\pi}{2}\right) = -1.$$

66. The most general form of boundary condition normally encountered in second-order boundary-value problems is a linear combination of the function and its derivatives at both ends of the region. Solve through finite difference approximations with four subintervals:

$$x'' - tx' + t^2x = t^3,$$
$$x(0) + x'(0) - x(1) + x'(1) = 3,$$
$$x(0) - x'(0) + x(1) - x'(1) = 2.$$

67. Repeat Exercise 63, but use the Runge–Kutta–Fehlberg method. The errors will not be proportional to h^2

68. Repeat Exercise 66, but use the modified Euler method.

69. Can a boundary-value problem be solved with a Taylor-series expansion of the function? If it can, use the Taylor-series technique for several of the above problems. If it cannot be used, provide an argument in support of this.

▶**70.** In solving a boundary-value problem with finite difference quotients, using smaller values for h improves the accuracy. Can one make h too small?

71. Compare the number of numerical operations used in Example 6.5 to get Tables 6.18 and 6.19.

Section 6.8

72. Consider the characteristic-value problem with k restricted to real values:

$$y'' - k^2y = 0, \quad y(0) = 0, \quad y(1) = 0.$$

a. Show analytically that there is no solution except the trivial solution $y = 0$.
b. Show, by setting up a set of difference equations corresponding to the differential equation with $h = 0.2$, that there are no real values for k for which a solution to the set exists.
c. Show, using the shooting method, that it is impossible to match $y(1) = 0$ for any real value of k [except if $y'(0) = 0$, which gives the trivial solution].

▶**73.** For the equation

$$y'' - 3y' + 2k^2y = 0, \quad y(0) = 0, \quad y(1) = 0,$$

find the principal eigenvalue and compare to $|k| = 2.46166$,

a. using $h = \frac{1}{2}$.
b. using $h = \frac{1}{3}$.
c. using $h = \frac{1}{4}$.
d. Assuming errors are proportional to h^2, extrapolate from parts (a) and (c) to get an improved estimate.

74. Using the principal eigenvalue, $k = 2.46166$, in Exercise 73, find y as a function of x over $[0, 1]$. This is the corresponding eigenfunction.

75. Parallel the computations of Exercise 73 to estimate the second eigenvalue. Compare to the analytical value of 4.56773.

76. Find the dominant eigenvalue and the corresponding eigenvector by the power method:

a. $\begin{bmatrix} 3 & 1 \\ 2 & 9 \end{bmatrix}$ b. $\begin{bmatrix} 2 & 3 \\ 6 & 5 \end{bmatrix}$ c. $\begin{bmatrix} 2 & 3 \\ 3 & -2 \end{bmatrix}$

d. $\begin{bmatrix} 6 & 2 & 0 \\ 2 & 4 & 1 \\ 0 & 1 & -1 \end{bmatrix}$ e. $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 2 & 2 & 1 \end{bmatrix}$

[In part (c), the two eigenvalues are equal but of opposite sign.]

77. For the two matrices

$$A = \begin{bmatrix} -5 & 2 & 1 \\ 1 & -9 & -1 \\ 2 & -1 & 7 \end{bmatrix},$$

$$B = \begin{bmatrix} -4+2i & -1 & -5i \\ -3 & 7+i & -i \\ 2 & -1 & 4-i \end{bmatrix},$$

a. Put bounds on the eigenvalues using Gerschgorin's theorem.
b. Can you tell from part (a) whether either of the matrices is singular?

78. Use the power method or its variations to find all of the eigenvalues and eigenvectors for the matrices of Exercise 77. For matrix B, do you need to use complex arithmetic?

▶79. Get the eigenvalues for matrix A in Exercise 77 from its characteristic polynomial. Then invert the matrix and show that the eigenvalues are reciprocals but the eigenvectors are the same. How do the two characteristic polynomials differ? Can you get the second polynomial directly from the first? Can you do all of this for matrix B?

80. Repeat Exercise 79, but use the power method to get the dominant eigenvalue. Then shift by that amount and get the next one. Finally, get the third from the trace of A.

81. Find three matrices that convert one of the below diagonal elements to zero for matrix A of Exercise 77.

82. Use the matrices of Exercise 81 successively to make one element below the diagonal of A equal to zero, then multiply that product and the inverse of the rotation matrix (which is easy to find because it is just its transpose). We keep the eigenvalues the same because the two multiplications are a similarity transformation.

Repeat this process until all elements below the diagonal are less than 1.0E-4. When this is done, compare the elements now on the diagonal to the eigenvalues of A obtained by iteration. (This will take many steps. You will want to write a short computer program to carry it out.)

▶83. Use similarity transformations to reduce the matrix to upper Hessenberg. (Do no column or row interchanges.)

$$C = \begin{bmatrix} 3 & -1 & 2 & 7 \\ 1 & 2 & 0 & -1 \\ 4 & 2 & 1 & 1 \\ 2 & -1 & -2 & 2 \end{bmatrix}$$

84. Repeat Exercise 83 but with row/column interchanges that maximize the magnitude of the divisors.

85. Repeat Exercise 82 after first converting to upper Hessenberg. How many fewer iterations are needed?

Applied Problems and Projects

APP1. The mass in Figure 6.8 moves horizontally on the frictionless bar. It is connected by a spring to a support located centrally below the bar. The unstretched length of the spring is $L = \sqrt{(10)} = 3.1623$ m (meters); the spring constant is $k = 100$ N/m (newtons per meter); the mass of the block is 3 kg. Let $x(t)$ be the distance from the center of the bar to the location of the block at time t. Clearly the equilibrium position of the block is at $x = 1.0$ m (or $x = -1.0$ m). Let $y_0 = \sqrt{10}$ m (the unstretched length of the spring). This second-order differential equation describes the motion:

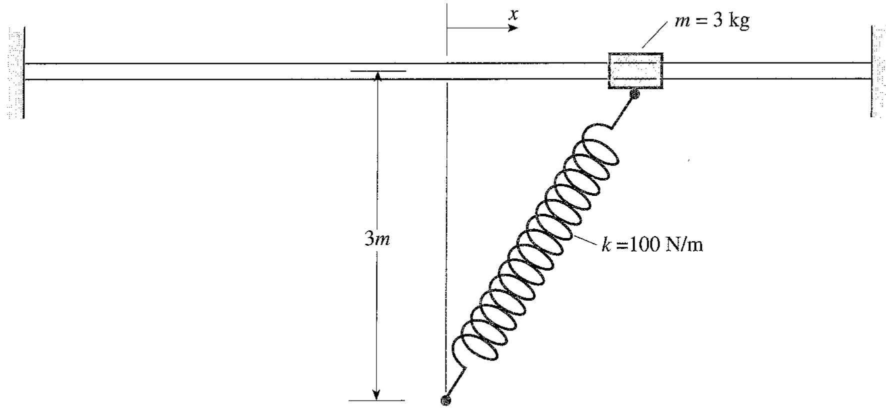

Figure 6.8

$$\frac{d^2x}{dt^2} = -\left(\frac{k}{m}\right)x\left(1 - \frac{y_0}{\sqrt{(x^2+9)}}\right).$$

a. Using both single-step and multistep methods, find the position of the block between $t = 0$ and $t = 10$ sec if $x_0 = 1.4$ and the initial velocity is zero.
b. Repeat part (a), but now with the spring stretched more at the start, $x_0 = 2.5$.
c. Use Maple and/or MATLAB to graph the motion for both parts (a) and (b). Compare your graphs to Figure 6.9.

APP2. The equation $y' = 1 + y^2$, $y(0) = 0$ has the solution $y = \tan(x)$. Use modified Euler method to compute values for $x = 0$ to $x = 1.6$ with a value for h small enough to obtain values that differ from the analytical by no more than ± 0.0005. What is the largest h-value to do this? $y\ (x)$ becomes infinite at $x = \pi/2$. What happens if you try to integrate y' beyond this point? Is there some way you can solve the equation numerically from $x = 0$ to $x = 2$?

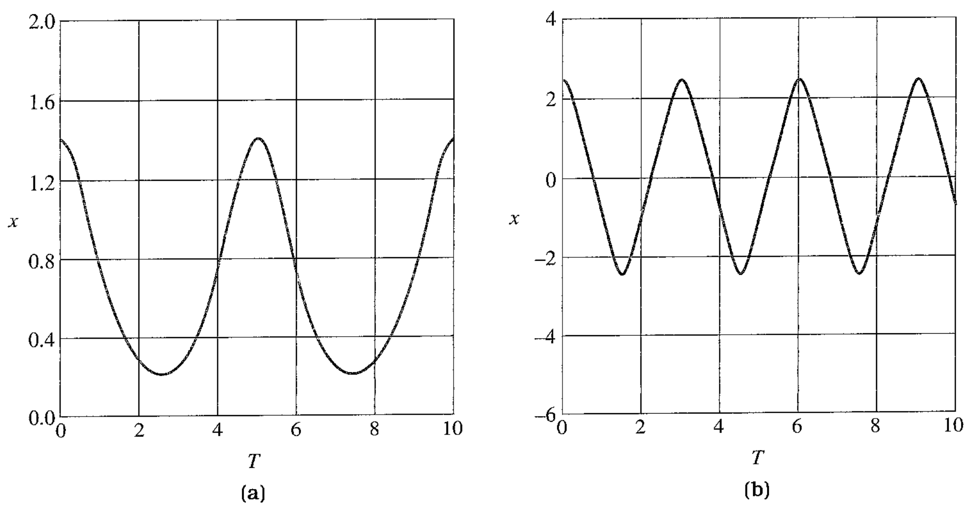

Figure 6.9

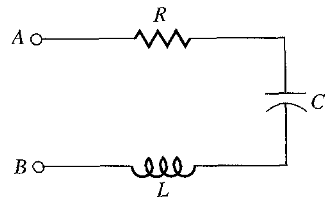

Figure 6.10

APP3. A nonlinear boundary-value problem is more difficult than a linear problem because many trials may be needed to get a good value for the initial slope. From three initial trials it should be possible to use a Muller's-type interpolation. Outline the steps of a program that will do this.

APP4. In an electrical circuit (Figure 6.10) that contains resistance, inductance, and capacitance (and every circuit does), the voltage drop across the resistance is iR (i is current in amperes, R is resistance in ohms), across the inductance it is L, (di/dt) (L is inductance in henries), and across the capacitance it is q/C (q is charge in the capacitor in coulombs, C is capacitance in farads). We then can write, for the voltage, difference between points A and B,

$$V_{AB} = L\frac{di}{dt} + Ri + \frac{q}{C}.$$

Differentiating with respect to t and remembering that $dq/dt = i$, we have a second-order differential equation;

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dV}{dt}.$$

If the voltage V_{AB} (which has previously been 0 V) is suddenly brought to 15 V (let us say, by connecting a battery across the terminals) and maintained steadily at 15 V (so $dV/dt = 0$), current will flow through the circuit. Use an appropriate numerical method to determine how the current varies with time between 0 and 0.1 sec if $C = 1000\ \mu$f, $L = 50$ mH, and $R = 4.7$ ohms; use Δt of 0.002 sec. Also determine how the voltage builds up across the capacitor during this time. You may want to compare the computations with the analytical solution.

APP5. Repeat App 4, but let the voltage source be a 60-Hz sinusoidal input:

$$V_{AB} = 15\sin(120\pi t).$$

How closely does the voltage across the capacitor resemble a sine wave during the last full cycle of voltage variation?

APP6. After the voltages have stabilized in APP4 (15 V across the capacitor), the battery is shorted so that the capacitor discharges through the resistance and inductor. Follow the current and the capacitor voltages for 0.1 sec, again with $\Delta t = 0.002$ sec. The oscillations of decreasing amplitude are called *damped oscillations*. If the calculations are repeated but with the resistance value increased, the oscillations will be damped out more quickly; at $R = 14.14$ ohms the oscillations should disappear; this is called *critical damping*. Perform numerical computations with values of R increasing from 4.7 to 22 ohms to confirm that critical damping occurs at 14.14 ohms.

APP7. Cooling fins are often welded to objects in which heat is generated to conduct the heat away, thus controlling the temperature. If the fin loses heat by radiation to the surroundings the rate of heat loss from the fin is proportional to the difference in fourth powers of the fin temperature and the surroundings, both measured in absolute degrees. The equation reduces to

$$d^2u/dx^2 = k(u^4 - T^4)$$

where u is the fin temperature, T is the surroundings temperature, and x is the distance along the fin. k is a constant. For a fin of given length L, this is not difficult to solve numerically if $u(0)$ and $u(L)$ are known. Solve for $u(x)$, the distribution of temperature along the fin, if $T = 300$, $u(0) = 450$, $u(20) = 350$, $k = 0.23$, utilizing any of the methods for a boundary-value problem. Use a value for h small enough to get temperatures accurate to 0.1 degree.

APP8. In APP7, suppose the fin is of infinite length and we can assume that $\lim (u(x)) = 0$ as $x \to \infty$. Can this problem be solved numerically? If so, get the solution for $u(x)$ between $x = 0$ and $x = 20$.

APP9. A Foucault pendulum is one free to swing in both the x- and y-directions. It is frequently displayed in science museums to exhibit the rotation of the earth, which causes the pendulum to swing in directions that continuously vary. The equations of motion are

$$\ddot{x} - 2\omega \sin \psi \dot{y} + k^2 x = 0,$$
$$\ddot{y} + 2\omega \sin \psi \dot{x} + k^2 y = 0,$$

when damping is absent (or compensated for). In these equations, the dots over the variable represent differentiation with respect to time. Here ω is the angular velocity of the earth's rotation (7.29×10^{-5} sec^{-1}), ψ is the latitude, $k^2 = g/\ell$ where ℓ is the length of the pendulum. How long will it take a 10-m-long pendulum to rotate its plane of swing by 45° at the latitude where you live? How long if located in Quebec, Canada?

APP10. Condon and Odishaw (1967) discuss Duffing's equation for the flux ϕ in a transformer. This nonlinear differential equation is

$$\ddot{\phi} + \omega_0^2 \phi + b\phi^3 = \frac{\omega}{N} E \cos \omega t.$$

In this equation, $E \sin \omega t$ is the sinusoidal source voltage and N is the number of turns in the primary winding, while ω_0 and b are parameters of the transformer design. Make a plot of ϕ versus t (and compare to the source voltage) if $E = 165$, $\omega = 120\pi$, $N = 600$, $\omega_0^2 = 83$, and $b = 0.14$. For approximate calculations, the nonlinear term $b\phi^3$ is sometimes neglected. Evaluate your results to determine whether this makes a significant error in the results.

APP11. Ethylene oxide is an important raw material for the manufacture of organic chemicals. It is produced by reacting ethylene and oxygen together over a silver catalyst. Laboratory studies gave the equation shown.

It is planned to use this process commercially by passing the gaseous mixture through tubes filled with catalyst. The reaction rate varies with pressure, temperature, and concentrations of ethylene and oxygen, according to this equation:

$$r = 1.7 \times 10^6 e^{-9716/T} \left(\frac{P}{14.7}\right) C_E^{0.328} C_O^{0.672},$$

where

r = reaction rate (units of ethylene oxide formed per lb of catalyst per hr),
T = temperature, °K (°C + 273),
P = absolute pressure (lb/in.^{2}),
C_E = concentration of ethylene,
C_O = concentration of oxygen.

Under the planned conditions, the reaction will occur, as the gas flows through the tube, according to the equation

$$\frac{dx}{dL} = 6.42r,$$

where

x = fraction of ethylene converted to ethylene oxide,
L = length of reactor tube (ft).

The reaction is strongly exothermic, so that it is necessary to cool the tubular reactor to prevent overheating. (Excessively high temperatures produce undesirable side reactions.) The reactor will be cooled by surrounding the catalyst tubes with boiling coolant under pressure so that the tube walls are kept at 225°C. This will remove heat proportional to the temperature difference between the gas and the boiling water. Of course, heat is generated by the reaction. The net effect can be expressed by this equation for the temperature change per foot of tube, where B is a design parameter:

$$\frac{dT}{dL} = 24{,}302r - B(T - 225).$$

For preliminary computations, it has been agreed that we can neglect the change in pressure as the gases flow through the tubes; we will use the average pressure of $P = 22$ lb/in.^{2} absolute. We will also neglect the difference between the catalyst temperature (which should be used to find the reaction rate) and the gas temperature. You are to compute the length of tubes required for 65% conversion of ethylene if the inlet temperature is 250°C. Oxygen is consumed in proportion to the ethylene converted; material balances show that the concentrations of ethylene and oxygen vary with x, the fraction of ethylene converted, as follows:

$$C_E = \frac{1 - x}{4 - 0.375x},$$

$$C_O = \frac{1 - 1.125x}{4 - 0.375x}.$$

The design parameter B will be determined by the diameter of tubes that contain the catalyst. (The number of tubes in parallel will be chosen to accommodate the quantities of materials flowing through the reactor.) The tube size will be chosen to control the maximum temperature of the reaction, as set by the minimum allowable value of B. If the tubes are too large in diameter (for which the value of B is small), the temperatures will run wild. If the tubes are *too small* (giving a large value to B), so much heat is lost that the reaction tends to be quenched. In your studies, vary B to find the least value that will keep the maximum temperature below 300°C. Permissible values for the parameter B are from 1.0 to 10.0.

In addition to finding how long the tubes must be, we need to know how the temperature varies with x and with the distance along the tubes. To have some indication of the controllability of the process, you are also asked to determine how much the outlet temperature will change for a 1°C change in the inlet temperature, using the value of B already determined.

APP12. An ecologist has been studying the effects of the environment on the population of field mice. Her research shows that the number of mice born each month is proportional to the number of females in the group and that the fraction of females is normally constant in any group. This implies that the number of births per month is proportional to the total population.

She has located a test plot for further research, which is a restricted area of semiarid land. She has constructed barriers around the plot so mice cannot enter or leave. Under the conditions of the experiment, the food supply is limited, and it is found that the death rate is affected as a result, with mice dying of starvation at a rate proportional to some power of the population. (She also hypothesizes that when the mother is undernourished, the babies have less chance for survival and that starving males tend to attack one another, but these factors are only speculation.)

The net result of this scientific analysis is the following equation, with N being the number of mice at time t (with t expressed in months). The ecologist has come to you for help in solving the equation; her calculus doesn't seem to apply.

$$\boxed{\frac{dN}{dt} = aN - BN^{1.7}, \qquad \text{with } B \text{ given by Table 6.20.}}$$

Table 6.20

t	B	t	B
0	0.0070	5	0.0013
1	0.0036	6	0.0028
2	0.0011	7	0.0043
3	0.0001	8	0.0056
4	0.0004		

As the season progresses, the amount of vegetation varies. The ecologist accounts for this change in the food supply by using a "constant" B that varies with the season.

If 100 mice were initially released into the test plot and if $a = 0.9$, estimate the number of mice as a function of t, for $t = 0$ to $t = 8$.

APP13. A certain chemical company produces a product that is a mixture of two ingredients, A and B. In order to ensure that the product is homogeneous, A and B are fed into a well-mixed tank that holds 100 gal. The desired product must contain two parts of A to one part of B within certain specifications. The normal flows of A and B into the tank are 4 and 2 gal/min. There is no volume change when these are mixed, so the outflow is 6 gal/min and the holding time in the tank is $100/6 = 16.66$ min. Due to an unfortunate accident, the flow of ingredient B is cut off and before this is noticed and corrected, the ratio of A to B in the tank has increased to 10 parts of A to 1 part of B. (There are still 100 gal in the tank.) Set up equations that give the ratio of A to B in the tank as a function of time after the flow of B has been restored to its normal value of 2 gal/min. How long will it take until the output from the tank reaches 2 parts A to 0.99 parts B? How much product is produced (and discarded because it is not up to specification) during this time? How would you suggest that this time to reach specification be reduced?