Example 2.8: Consider the nonlinear system

$$\dot{x}_1 = g(x_2) + 4x_1 x_2^2$$

$$\dot{x}_2 = h(x_1) + 4x_1^2 x_2$$

Since

$$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = 4(x_1^2 + x_2^2)$$

which is always strictly positive (except at the origin), the system does not have any limit cycles anywhere in the phase plane.   □

The above three theorems represent very powerful results. It is important to notice, however, that they have no equivalent in higher-order systems, where exotic asymptotic behaviors other than equilibrium points and limit cycles can occur.

2.7 Summary

Phase plane analysis is a graphical method used to study second-order dynamic systems. The major advantage of the method is that it allows visual examination of the global behavior of systems. The major disadvantage is that it is mainly limited to second-order systems (although extensions to third-order systems are often achieved with the aid of computer graphics). The phenomena of multiple equilibrium points and of limit cycles are clearly seen in phase plane analysis. A number of useful classical theorems for the prediction of limit cycles in second-order systems are also presented.

2.8 Notes and References

Phase plane analysis is a very classical topic which has been addressed by numerous control texts. An extensive treatment can be found in [Graham and McRuer, 1961]. Examples 2.2 and 2.3 are adapted from [Ogata, 1970]. Examples 2.5 and 2.6 and section 2.6 are based on [Hsu and Meyer, 1968].

2.9 Exercises

2.1   Draw the phase portrait and discuss the properties of the linear, unity feedback control system of open-loop transfer function

$$G(p) = \frac{10}{p\,(1 + 0.1\,p)}$$

2.2 Draw the phase portraits of the following systems, using isoclines

(a) $\quad \ddot{\theta} + \dot{\theta} + 0.5\,\theta = 0$

(b) $\quad \ddot{\theta} + \dot{\theta} + 0.5\,\theta = 1$

(c) $\quad \ddot{\theta} + \dot{\theta}^2 + 0.5\,\theta = 0$

2.3 Consider the nonlinear system

$$\dot{x} = y + x\,(x^2 + y^2 - 1)\,\sin\frac{1}{x^2 + y^2 - 1}$$

$$\dot{y} = -x + y\,(x^2 + y^2 - 1)\,\sin\frac{1}{x^2 + y^2 - 1}$$

Without solving the above equations explicitly, show that the system has infinite number of limit cycles. Determine the stability of these limit cycles. (*Hint*: Use polar coordinates.)

2.4 The system shown in Figure 2.10 represents a satellite control system with rate feedback provided by a gyroscope. Draw the phase portrait of the system, and determine the system's stability.
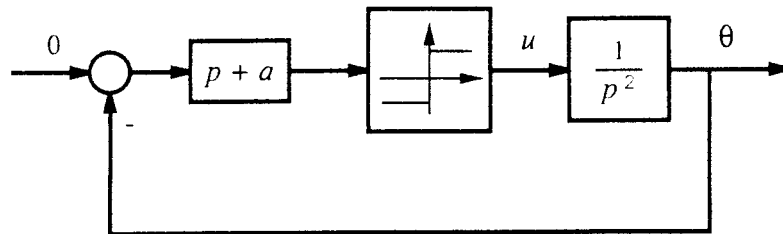

Figure 2.10 : Satellite control system with rate feedback