3.9 Exercises

3.1 The norm used in the definitions of stability need not be the usual Euclidian norm. If the state-space is of finite dimension n (*i.e.*, the state vector has n components), stability and its type are independent of the choice of norm (all norms are "equivalent"), although a particular choice of norm may make analysis easier. For $n = 2$, draw the unit balls corresponding to the following norms:

(i) $\| \mathbf{x} \|^2 = (x_1)^2 + (x_2)^2$ (Euclidian norm)

(ii) $\| \mathbf{x} \|^2 = (x_1)^2 + 5\,(x_2)^2$

(iii) $\| \mathbf{x} \| = |x_1| + |x_2|$

(iv) $\| \mathbf{x} \| = \mathrm{Sup}(\,|x_1|\,,\,|x_2|\,)$

Recall that a ball $\mathbf{B}(\mathbf{x}_o\,, R)$, of center $\mathbf{x}_o$ and radius R, is the set of $\mathbf{x}$ such that $\| \mathbf{x} - \mathbf{x}_o \| \leq R$, and that the unit ball is $\mathbf{B}(\mathbf{0}, 1)$.

3.2 For the following systems, find the equilibrium points and determine their stability. Indicate whether the stability is asymptotic, and whether it is global.

(a) $\dot{x} = -x^3 + \sin^4 x$

(b) $\dot{x} = (5 - x)^5$

(c) $\ddot{x} + \dot{x}^5 + x^7 = x^2 \sin^8 x \cos^2 3x$

(d) $\ddot{x} + (x-1)^4 \dot{x}^7 + x^5 = x^3 \sin^3 x$

(e) $\ddot{x} + (x-1)^2 \dot{x}^7 + x = \sin(\pi x/2)$

3.3 For the Van der Pol oscillator of Example 3.3, demonstrate the existence of a limit cycle using the linearization method.

3.4 This exercise, adapted from [Hahn, 1967], provides an example illustrating the motivation of the radial unboundedness condition in Theorem 3.3. Consider the second-order system

$$\dot{x}_1 = -\frac{6x_1}{z^2} + 2x_2$$

$$\dot{x}_2 = -\frac{2(x_1 + x_2)}{z_2}$$

with $z = 1 + x_1{}^2$. On the hyperbola $x_2^h = 2/(x_1 - \sqrt{2}\,)$, the system trajectory slope is