$$\frac{\dot{x}_2}{\dot{x}_1} = \frac{-1}{1 + 2^{3/2}x_1 + 2x_1^{\,2}}$$

while the slope of the hyperbola is

$$\frac{dx_2^{\,h}}{dx_1} = \frac{-1}{1 - 2^{3/2}x_1 + x_1^{\,2}/2}$$

Note that for $x_1 > \sqrt{2}$, the first expression is larger than the second, implying that the trajectories cannot cut the branch of the hyperbola which lies in the first quadrant, in the direction toward the axes (since on the hyperbola we have $\dot{x}_1 > 0$ if $x_1 > \sqrt{2}$). Thus, there are trajectories which do not tend toward the origin, indicating the lack of global asymptotic stability. Use the scalar function

$$V(\mathbf{x}) = \frac{x_1^{\,2}}{z} + x_2^{\,2}$$

to analyze the stability of the above system.

3.5 Determine regions of attraction of the pendulum, using as Lyapunov functions the pendulum's total energy, and the modified Lyapunov function of page 67. Comment on the two results.

3.6 Show that given a constant matrix $\mathbf{M}$ and any time-varying vector $\mathbf{x}$, the time-derivative of the scalar $\mathbf{x}^T\,\mathbf{M}\,\mathbf{x}$ can be written

$$\frac{d}{dt}\mathbf{x}^T\,\mathbf{M}\,\mathbf{x} = \mathbf{x}^T\,(\mathbf{M} + \mathbf{M}^T)\,\dot{\mathbf{x}} = \dot{\mathbf{x}}^T\,(\mathbf{M} + \mathbf{M}^T)\,\mathbf{x}$$

and that, if $\mathbf{M}$ is symmetric, it can also be written

$$\frac{d}{dt}\mathbf{x}^T\,\mathbf{M}\,\mathbf{x} = 2\,\mathbf{x}^T\,\mathbf{M}\,\dot{\mathbf{x}} = 2\,\dot{\mathbf{x}}^T\,\mathbf{M}\,\mathbf{x}$$

3.7 Consider an $n \times n$ matrix $\mathbf{M}$ of the form $\mathbf{M} = \mathbf{N}^T\mathbf{N}$, where $\mathbf{N}$ is a $m \times n$ matrix. Show that $\mathbf{M}$ is *p.d.* if, and only if, $m \geq n$ and $\mathbf{N}$ has full rank.

3.8 Show that if $\mathbf{M}$ is a symmetric matrix such that

$$\forall\,\mathbf{x}\,,\;\; \mathbf{x}^T\mathbf{M}\mathbf{x} = 0$$

then $\mathbf{M} = \mathbf{0}$.

3.9 Show that if symmetric *p.d.* matrices $\mathbf{P}$ and $\mathbf{Q}$ exist such that

$$\mathbf{A}^T\,\mathbf{P} + \mathbf{P}\,\mathbf{A} + 2\lambda\mathbf{P} = -\mathbf{Q}$$

then all the eigenvalues of $\mathbf{A}$ have a real part strictly less than $-\lambda$. (Adapted from [Luenberger, 1979].)

3.10 Consider the system

$$\mathbf{A}_1\,\ddot{\mathbf{y}} + \mathbf{A}_2\,\dot{\mathbf{y}} + \mathbf{A}_3\,\mathbf{y} = \mathbf{0}$$

where the $2n \times 1$ vector $\mathbf{x} = [\mathbf{y}^T \ \dot{\mathbf{y}}^T]^T$ is the state, and the $n \times n$ matrices $\mathbf{A}_j$ are all symmetric positive definite. Show that the system is globally asymptotically stable, with $\mathbf{0}$ as a unique equilibrium point.

3.11 Consider the system

$$\dot{\mathbf{x}} = \mathbf{A}\,\mathbf{x} \qquad y = \mathbf{c}^T\mathbf{x}$$

Use the invariant set theorem to show that if the system is observable, and if there exists a symmetric *p.d.* matrix $\mathbf{P}$ such that

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\,\mathbf{A} \ = \ -\,\mathbf{c}\,\mathbf{c}^T$$

then the system is asymptotically stable.

Can the result be derived using the direct method? (Adapted from [Luenberger, 1979].)

3.12 Use Krasovskii's theorem to justify Lyapunov's linearization method.

3.13 Consider the system

$$\dot{x} = 4x^2 y - f_1(x)\,(x^2 + 2y^2 - 4)$$

$$\dot{y} = -\,2x^3 - f_2(y)\,(x^2 + 2y^2 - 4)$$

where the continuous functions f_1 and f_2 have the same sign as their argument. Show that the system tends towards a limit cycle independent of the explicit expressions of f_1 and f_2.

3.14 The second law of thermodynamics states that the entropy of an isolated system can only increase with time. How does this relate to the notion of a Lyapunov function?