found. The statement of Theorem 4.9 follows that in [Bodson, 1986]. Lemma 4.2 and its proof are from [Popov, 1973]. An extensive study of absolute stability problems from a frequency-domain perspective is contained in [Narendra and Taylor, 1973], from which the definitions and theorems on positive real functions are adapted. A more recent description of positive real functions and their applications in adaptive control can be found in [Narendra and Annasswamy, 1989]. The Bellman-Gronwall lemma and its proof are adapted from [Hsu and Meyer, 1968]. The definition and theorem on total stability are based on [Hahn, 1965]. Example 4.23 is adapted from [Desoer, *et a.*, 1965].

Passivity theory (see [Popov, 1973; Desoer and Vidyasagar, 1975]) is presented in a slightly unconventional form. Passivity interpretations of adaptive control laws are discussed in [Landau, 1979]. The reader is referred to [Vidyasagar, 1978] for a detailed discussion of absolute stability. The circle criterion and its extensions to non-autonomous systems were derived by [Narendra and Goldwyn, 1964; Sandberg, 1964; Tsypkin, 1964; Zames, 1966].

Other important robustness analysis tools include singular perturbations (see, *e.g.*, [Kokotovic, *et al.*, 1986]) and averaging (see, *e.g*, [Hale, 1980; Meerkov, 1980]).

Relations between the existence of Lyapunov functions and the existence and unicity of solutions of nonlinear differential equations are discussed in [Yoshizawa, 1966, 1975].

4.13   Exercises

4.1   Show that, for a non-autonomous system, a system trajectory is generally not an invariant set.

4.2   Analyze the stability of the dynamics (corresponding to a mass sinking in a viscous liquid)

$$\dot{v} + 2a|v|v + bv = c \qquad a > 0, b > 0$$

4.3   Show that a function $V(\mathbf{x}, t)$ is radially unbounded if, and only if, there exists a class-K function ϕ such that

$$V(\mathbf{x}, t) \geq \phi(\|\mathbf{x}\|)$$

where the function ϕ satisfies

$$\lim_{\mathbf{x} \to \infty} \phi(\|\mathbf{x}\|) = \infty$$

4.4   The performance of underwater vehicles control systems is often constrained by the "unmodeled" dynamics of the thrusters. Assume that one decides to explicitly account for thruster dynamics, based on the model

$$\dot{\omega} = -\alpha_1 \omega |\omega| + \alpha_2 \tau \qquad \alpha_1 > 0, \alpha_2 > 0$$

$$u = b \omega |\omega| \qquad b > 0$$

where τ is the torque input to the propeller, ω is the propeller's angular velocity, and u is the actual thrust generated.

Show that, for a *constant* torque input τ_o, the steady-state thrust is proportional to τ_o (which is consistent witht the fact that thruster dynamics is often treated as "unmodeled").

Assuming that the coefficients α_i and b in the above model are known with good accuracy, design and discuss the use of a simple "open-loop" observer for u (given an arbitrary time-varying torque input τ) in the absence of measurements of ω. (Adapted from [Yoerger and Slotine, 1990].)

4.5 Discuss the similarity of the results of section 4.2.2 with Krasovskii's theorem of section 3.5.2.

4.6 Use the first instability theorem to show the instability of the vertical-up position of a pendulum.

4.7 Show explicitly why the linear time-varying system defined by (4.18) does not satisfy the sufficient condition (4.19).

4.8 Condition (4.19) on the eigenvalues of $A(t) + A^T(t)$ is only, of course, a *sufficient* condition. For instance, show that the linear time-varying system associated with the matrix

$$A(t) = \begin{bmatrix} -1 & e^{t/2} \\ 0 & -1 \end{bmatrix}$$

is globally asymptotically stable.

4.9 Determine whether the following systems have a stable equilibrium. Indicate whether the stability is asymptotic, and whether it is global.

(a)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -10 & e^{3t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & 2\sin t \\ 0 & -(t+1) \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(c)
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -1 & e^{2t} \\ 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

4.10 If a differentiable function f is lower bounded and decreasing ($\dot{f} \leq 0$), then it converges to a limit. However, $\dot{f}$ does not necessarily converge to zero. Derive a counter-example. (*Hint:* You may

use for $-\dot{f}$ a function that peaks periodically, but whose integral is finite.)

4.11 (a) Show that if a function f is bounded and uniformly continuous, and there exists a positive definite function $F(f, t)$ such that

$$\int_{0}^{\infty} F(f(t), t)\, dt < \infty$$

then $f(t)$ tends to zero as $t \to \infty$.

(b) For a given autonomous nonlinear system, consider a Lyapunov function V in a ball $\mathbf{B}_R$, and let ϕ be a scalar, differentiable, strictly monotonously increasing function of its scalar argument. Show that $[\phi(V) - \phi(0)]$ is also a Lyapunov function for the system (distinguish the cases of stability and of asymptotic stability). Suggest extensions to non-autonomous systems.

4.12 Consider a scalar, lower bounded, and twice continuously differentiable function $V(t)$ such that

$$\forall\, t \geq 0, \quad \dot{V}(t) \leq 0$$

Show that, for any $t \geq 0$,

$$\dot{V}(t) = 0 \quad \Rightarrow \quad \ddot{V}(t) = 0$$