Problem 1: (20 points)

A system with input $u(t)$ and output $y(t)$ has the transfer function representation

$$\frac{Y(s)}{U(s)} = \frac{10s+20}{s^2-100}$$

1. *(2 points) Sketch the s-plane pole-zero plot of the transfer function.*
2. *(5 points) Find the ODE representation of the system and express your answer in the standard form*

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \cdots + a_0 y = b_m \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \cdots + b_0 u$$

3. *(3 points) If applicable, what is the DC gain of the system? If the concept of DC gain is not applicable to this system, explain why in one or two short sentences.*
4. *(5 points) Draw an SFG graph for this system.*
5. *(5 points) By assigning a state variable to the output of each integrator, write, in matrix form, the state equations.*

Problem 2: (25 points)

Consider the state-space model

$$\begin{pmatrix} \dot{\bar{x}}_1 \\ \dot{\bar{x}}_2 \end{pmatrix} = \begin{pmatrix} 10 & 0 \\ 0 & -10 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

$$y = \begin{pmatrix} 6 & 4 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \end{pmatrix}$$

1. *(10 points) Draw an SFG representation of the system.*
2. *(12 points) Using the SFG, find the transfer function representation of the system.*
3. *(3 points) Compare the transfer function obtained in part 2 to that in problem 1; they should be identical. Is it possible to represent the system in problem 2 using the state space model derived in problem 1? Based on your answer, is the state-space representation of a given physical system unique?*

Problem 3: (25 points)

Consider a LTI SISO system with the ODE representation

$$\ddot{y} + 3\dot{y} + 2y = u$$

1. *(8 points) Using Laplace transform methods, calculate the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$ for the initial condition $y(0)=1, \dot{y}(0)=-1$ and the input $u(t)=e^{-3t}u_0(t)$, where $u_0(t)$ is the unit-step function.*
2. *(4 points) Find a state-space representation using the sate variable assignment $x_1=\dot{y}$ and $x_2=y$. What is the initial state vector $x(0)$?*
3. *(5 points) Compare the state-transition matrix $\Phi(t)$. Check your answer by showing that $\Phi(0)=I$.*
4. *(8 points) Using the state-transition matrix, calculate the zero-input response $y_{zi}(t)$ and the zero-state response $y_{zs}(t)$ for the initial conditions and the input provided in part 1.*

Problem 4: (20 points)

In addition to the external force $u(t)$ *, the ass* $M = 1\ Kg$ *shown in figure 1 experiences a restoring force* $(1+y^2)y$ *provided by a* hardening *spring. Beyond a certain displacement, a small change in spring length produces a large restoring force.*

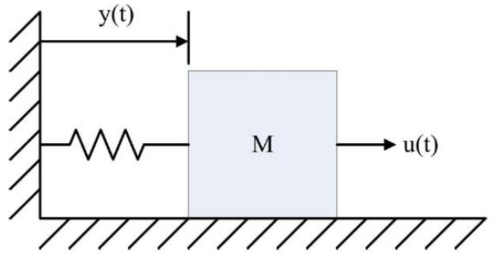

Figure 1: Mass-spring mechanical system.

Newton's second law provide the following nonlinear equation of motion for the system

$\ddot{y} = -(1+y^2)y + u(t)$

1. *(5 points) Using* $x_1 = y$ *and* $x_2 = \dot{y}$ *as state variables, show that the equations of motion can also be described by a nonlinear state-space model of the form*

$$\begin{aligned}\dot{x}_1 &= f_1(x_1, x_2, u)\\ \dot{x}_2 &= f_2(x_1, x_2, u)\\ y &= g_1(x_1, x_2)\end{aligned}$$

   Specify the functions f_1, f_2 *and* g .

2. *(5 points) A constant external force* $u(t) = u_0 = 2N$ *is applied to the system. Find the physically meaningful static equilibrium state* x_e *for the system (the displacement y must be real and positive).* ***Hint:*** *Use the MATLAB command roots.*
3. *(10 points) Find a linear state-space model that describes the dynamic behavior of the mass-spring system for small perturbations from the static equilibrium state* x_e *and constant input* u_0 *. Let* $x = x_e + \delta x$ *and* $u = u_0 + \delta u$ *, and for the state-space representation*

$$\begin{aligned}\delta\dot{x} &= F\,\delta x + G\,\delta u\\ \delta y &= H\,\delta x,\end{aligned}$$

   Determine the matrices $F, G,$ *and* H.

Problem 5: (5 points)

A system is represented by form $\dot{x} = Ax + Bu$ *and* $y = Cx + Du.$ *where*

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

(a) *Find the matrix* $\phi(t)$.

(b) *For the initial conditions x1(0)=x2(0)=1, find x(t).*

Problem 6: (5 points)

For system described by differential equation, we define state variable as $x_1 = y, x_2 = \dot{y}$. *Find* g_1 *and* g_2 *so that state feedback with equation* $u(t)=r(t)- g_1x_1(t)- g_2x_2(t)$ *transfers poles of system to s=-2 and s=-3.*

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = u(t)$$