Problem 1: (20 points)

This problem reveals the relationship between the phase margin and dimensionless damping ratio for a second-order closed-loop system. Consider the closed-loop system in Figure 1 where the plant transfer function is

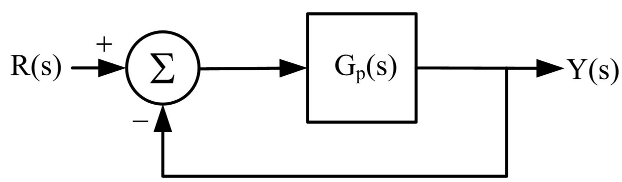

Fig.1

1. (2 points) Show that:

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

2. (7 points) Define the gain crossover frequency ω_{gc} as the frequency at which

$$\left|G(j\omega_{gc})\right| = 1.$$

   Show that the gain crossover frequency satisfies

$$\omega_{gc} = \omega_n\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}$$

3. (7 points) By considering a sketch of the polar plot of $G(j\omega)$, show that the phase margin (PM) is

$$PM = \tan^{-1}\frac{2\zeta}{\sqrt{\sqrt{1+4\zeta^4} - 2\zeta^2}}$$

4. (4 points) When the phase margin is less than $70°$, the following approximation is frequently used

$$\zeta \approx \frac{PM}{100},$$

   where PM is expressed in degrees. Verify that this approximation is valid by using MATLAB to plot phase margin (PM) versus ζ for $0 \le \zeta \le 1$ using the exact expression in part 3 and the above approximation.

Problem 2: (20 points)

The closed-loop system in Figure 2 has a loop transfer function whose Nyquist plot is shown in Figure 3 for unity proportional control K = 1.

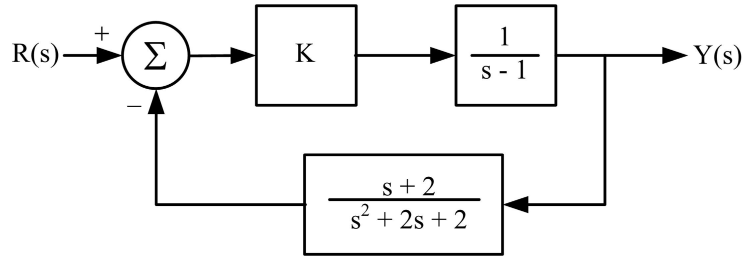

Fig. 2

Fig. 3

1. (5 points) What is the loop transfer function, and how many poles of the loop transfer function lie in the right-half plane?
2. (5 points) Using the Nyquist plot, for what range of K is the closed-loop system stable?
3. (5 points) Using the Nyquist plot, for what range of K is there one closed-loop pole in the right half of the s-plane. For what range of K are there two closed-loop poles in the right-half plane?
4. (5 points) Verify your answers in parts 2 and 3 by using the Routh-Hurwitz stability criterion.

Problem 3: (20 points)
Consider the closed-loop system in Figure 4.

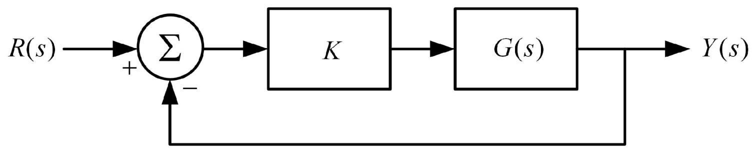

Fig. 4

For each system listed below:

- Neatly sketch by hand the root loci as the proportional gain is varied from zero towards infinity. Be sure to specify breakaway and breakin points, asymptotes, arrival and departure angles, and imaginary-axis crossings, if any. Check your answer using the MATLAB command **rlocus(num,den)**. Attach a copy of the MATLAB root locus plot along with your hand sketch.
- For $K \geq 0$, specify the range of K, if any, that the system is BIBO stable.

Problem 4: (20 points)

The feedback control system in Figure 5 utilizes a controller transfer function $G_c(s)$ in cascade with the plant transfer function
The zero is located at

where the nominal DC gain and time constant of the plant are $K = 10$ and $\tau = 1$, respectively. Denote the transfer function of the closed-loop system as

Fig.5

1. (3 points) Find the sensitivity $S_K^{G_p}$ of the open-loop plant transfer function to K and specify the value of the sensitivity $S_K^{G_p}$ at DC.
2. (5 points) A control engineer specifies the use of a proportional (P) controller

   Using P control, find the sensitivity S_K^G of the closed-loop transfer function to the open-loop DC gain K, and choose the value of K_p so that the DC gain of the closed-loop system is ten times less sensitive than the DC gain of the open-loop system to variations in K.
3. (5 points) The controller engineer asserts that the proportional-plus-integral (PI) controller

   Will null the sensitivity of the closed-loop DC gain to variations in K. Using PI control, find the sensitivity S_K^G of the closed-loop transfer function to the open-loop DC gain K. Under what constraints is the control engineer's assertion correct?
4. (7 points) Once again consider the closed-loop system that uses P control and the resulting sensitivity S_K^G function derived in part 2. Determine the sensitivity S_τ^G of the closed-loop transfer function to the plant time constant τ, and compare the two sensitivity functions by carefully sketching $\left|S_K^G(j\omega)\right|$ and $\left|S_\tau^G(j\omega)\right|$ as a function of ω. Using your sketch, at low frequencies is the closed-loop transfer function more sensitive to variations in the DC gain or the time constant of the open-loop system?