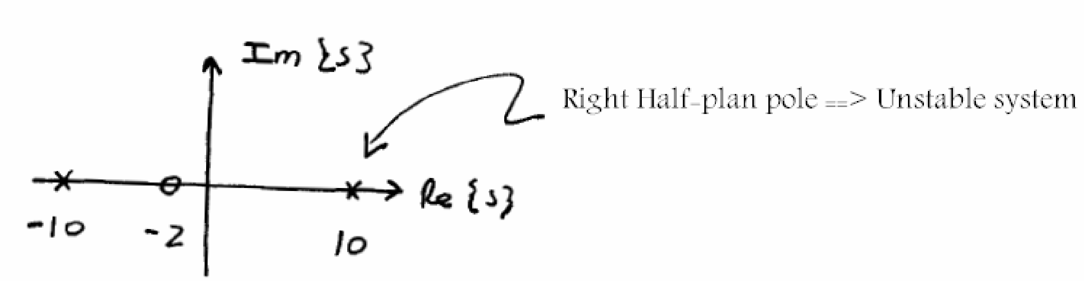
Solution of Problem 1:

1. $\frac{Y(s)}{U(s)} = \frac{10s+20}{s^2-100}$, Zeros satisfy $10s+20=0 \;\Rightarrow\; z_1=-2$

Poles satisfy $s^2-100=0 \;\Rightarrow\; p_1=-10,\; p_2=+10$



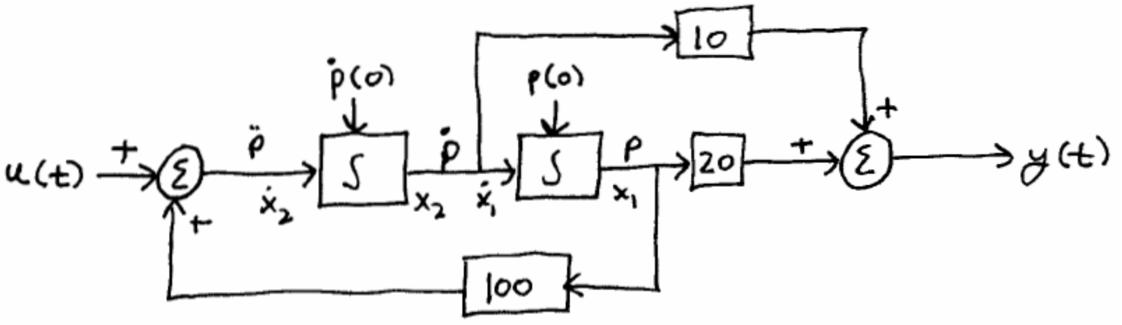
2. Cross multiplying,

$$Y(s)\{s^2-100\} = U(s)\{10s+20\}$$
$$\ddot{y} - 100y = 10\dot{u} + 20u$$

3. A system with a pole in the right half-plane is not stable, and so the concept of DC gain is not applicable. A constant input will not result in a finite, constant output.

4.

$$\frac{Y(s)}{U(s)} = \frac{P(s)}{U(s)}\frac{Y(s)}{P(s)} = \frac{1}{s^2-100}\{10s+20\}$$
$$P(s)\left[s^2-100\right] = U(s) \;\Rightarrow\; \ddot{p} = 100p + u$$
$$Y(s) = [10s+20]P(s) \;\Rightarrow\; y = 10\dot{p} + 20p$$



5. Choosing $x_1 = p$ *and* $x_2 = \dot{p}$ as shown above,

$$\dot{x}_1 = x_2 \quad \dot{x}_2 = 100x_1 + u \quad y = 20x_1 + 10x_2$$

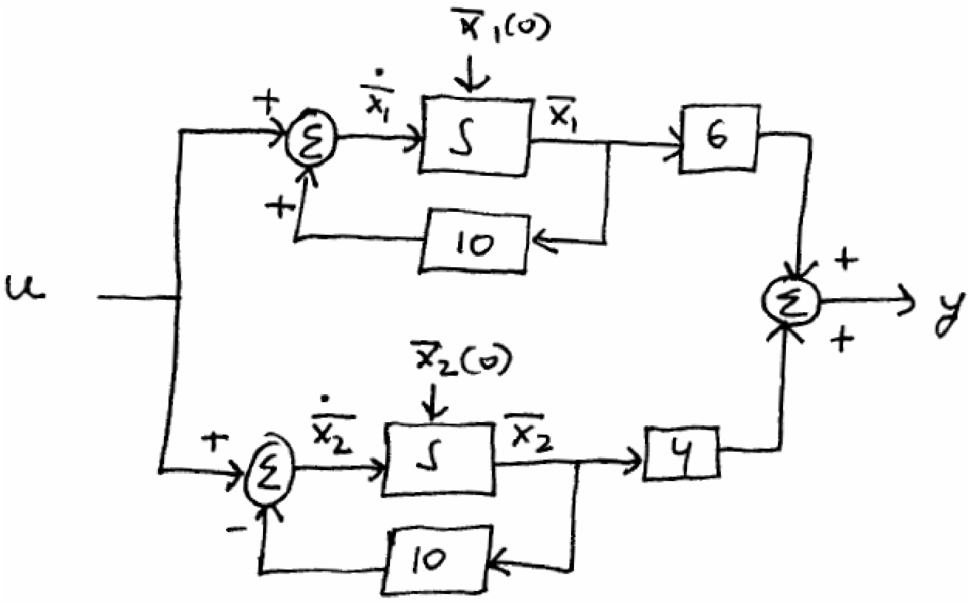
$$\dot{X} = \begin{pmatrix} 0 & 1 \\ 100 & 0 \end{pmatrix} X + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u \qquad Y = \begin{pmatrix} 20 & 10 \end{pmatrix} X$$

Solution of Problem 2:

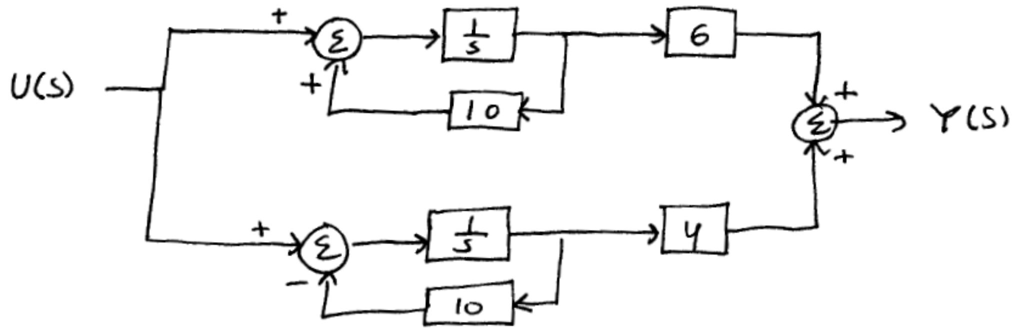
1.

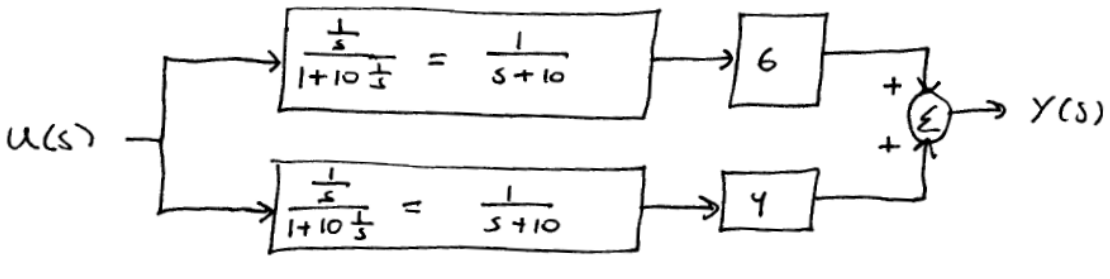
$$\begin{pmatrix}\dot{\bar{x}}_1\\ \dot{\bar{x}}_2\end{pmatrix}=\begin{pmatrix}10 & 0\\ 0 & -10\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}+\begin{pmatrix}1\\ 1\end{pmatrix}u \qquad Y=\begin{pmatrix}6 & 4\end{pmatrix}\begin{pmatrix}\bar{x}_1\\ \bar{x}_2\end{pmatrix}$$

$$\dot{\bar{x}}_1=10\bar{x}_1+u \qquad \dot{\bar{x}}_2=-10\bar{x}_2+u \qquad y=6\bar{x}_1+4\bar{x}_2$$



2. Redraw the block diagram in the s-domain, and because we are finaly a transfer function, set the initial conditions to zero:





$$Y(s)=\left[\frac{6}{s-10}+\frac{4}{s+10}\right]U(s)$$

$$\frac{Y(s)}{U(s)}=\frac{6(s+10)+4(s-10)}{(s-10)(s+10)}\quad\Rightarrow\quad\frac{Y(s)}{U(s)}=\frac{10s+20}{s^2-100}$$

3. The transfer function in part 2 is identical to that in problem 1 part 2. The state space models in problem 1 represent the same physical system. As a result, these two problems show that *state-space models are not unique*. As will be seen in future, a given physical system has an infinite number of state-space representations.

Solution of Problem 3:

1. The laplace transform of both sides of the ODE yields

$$L\{\ddot{y}+3\dot{y}+2y\}=L\{u\}$$

$$s^2-sy(0)-\dot{y}(0)+3Y(s)s-3y(0)+2Y(s)=u(s)$$

$$(s^2+3s+2)Y(s)=u(s)+sy(0)+\dot{y}(0)+3y(0)$$

$$Y(s)=\frac{1}{s^2+3s+2}u(s)+\frac{sy(0)+\dot{y}(0)+3y(0)}{s^2+3s+2}=L\{y_{zs}(t)\}+L\{y_{zi}(t)\}$$

As $u(t)=e^{-3t}u_0(t),\ u(s)=\dfrac{1}{s+3}$. Using this result with $y(0)=1\ and\ \dot{y}(0)=-1$ gives

$$Y_{zs}(s)=\frac{1}{(s^2+3s+2)(s+3)}\ \text{ and }\ Y_{zi}(s)=\frac{s+2}{s^2+3s+2}$$

Use a partial fraction expansion to find $y_{zs}(t)$ and $y_{zi}(t)$.

$$y_{zs}(t)=\frac{1}{2}e^{-t}u_0(t)-e^{-2t}u_0(t)+\frac{1}{2}e^{-3t}u_0(t)$$

$$y_{zi}(t)=e^{-t}u_0(t)$$

2. If we set $x_1=\dot{y}$ and $x_2=y$, then from the ODE

$$\ddot{y}=-3\dot{y}-2y-u$$

$$\dot{x}_1=-3x_1-2x_2+u\quad and\quad \dot{x}_2=x_1$$

The resulting state-space model is

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-3 & -2\\ 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 0\end{bmatrix}u$$

$$Y=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

As $y(0)=1 \ and \ \dot{y}(0)=-1, \quad X(0)=\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}=\begin{bmatrix} \dot{y}(0) \\ y(0) \end{bmatrix}=\begin{bmatrix} -1 \\ 1 \end{bmatrix}$

3.

$\Phi(t)=e^{At}u_0(t)=L^{-1}\left\{(sI-A)^{-1}\right\}$

$(sI-A)^{-1}=\dfrac{1}{s^2+3s+2}\begin{pmatrix} s & -2 \\ 1 & s+3 \end{pmatrix}$

Find the inverse laplace transforms using partial fraction expansions

$\Phi_{11}(s)=\dfrac{s}{(s+1)(s+2)}=\dfrac{-1}{s+1}+\dfrac{2}{s+2}, \qquad \Phi_{11}(t)=-e^{-t}u_0(t)+2e^{-2t}u_0(t)$

$\Phi_{21}(s)=\dfrac{1}{(s+1)(s+2)}=\dfrac{1}{s+1}-\dfrac{1}{s+2}, \qquad \Phi_{21}(t)=e^{-t}u_0(t)-e^{-2t}u_0(t)$

3. Note that $\Phi_{12}(s)=-2\Phi_{21}(s)$, and so

$\Phi_{12}(t)=-2\Phi_{21}(t)=-2e^{-t}u_0(t)+2e^{-2t}u_0(t)$

Also observe that

$\Phi_{22}(s)=\Phi_{11}(s)+3\Phi_{21}(s)$

And so

$\Phi_{22}(t)=\Phi_{11}(t)+3\Phi_{21}(t)=2e^{-t}u_0(t)-e^{-2t}u_0(t)$

Combining the results

$\Phi(t)=\begin{bmatrix} 2e^{-2t}-e^{-t} & 2e^{-2t}-2e^{-t} \\ e^{-t}-e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}u_0(t)$

As a check, $\Phi(0)=I$ as expected.

4. $X_{zi}(t)=\Phi(t)X(0)$ and $Y_{zi}(t)=CX_{zi}(t)$. It follows that

$Y_{zi}(t)=C\Phi(t)X(0)=\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 2e^{-2t}-e^{-t} & 2e^{-2t}-2e^{-t} \\ e^{-t}-e^{-2t} & 2e^{-t}-e^{-2t} \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix}u_0(t)=e^{-t}u_0(t) \quad \Rightarrow$ matches answer in part 1

$X_{zs}(t)=\int_0^t \Phi(t-\tau)Bu(\tau)d\tau \ , \quad Y_{zs}(t)=CX_{zs}(t)$

$C\Phi(t-\tau)B=\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \Phi_{11}(t-\tau) & \Phi_{12}(t-\tau) \\ \Phi_{21}(t-\tau) & \Phi_{22}(t-\tau) \end{pmatrix}\begin{pmatrix} 1 \\ 0 \end{pmatrix}=\begin{pmatrix} 0 & 1 \end{pmatrix}\begin{pmatrix} \Phi_{11}(t-\tau) \\ \Phi_{21}(t-\tau) \end{pmatrix}=\Phi_{21}(t-\tau)$

$Y_{zs}(t)=\int_0^t \Phi_{21}(t-\tau)u(\tau)d\tau=e^{-t}\left(\dfrac{1}{2}-\dfrac{1}{2}e^{-2t}\right)-e^{-2t}\left(1-e^{-t}\right) \qquad t\geq 0$

$Y_{zs}(t) = \left(\frac{1}{2}e^{-t} - e^{-2t} + \frac{1}{2}e^{-3t}\right)u_0(t)$, This result matches the answer in part 1.

Solution of Problem 4:

1. $\ddot{y} = -(1+y^2)y + u(t)$. Let $x_1 = y \ and \ x_2 = \dot{y}$, then

$$\left.\begin{aligned} \dot{x}_1 = x_2 \qquad\qquad\qquad &= f_1(x_1, x_2, u) \\ \dot{x}_2 = -(1+x_1^2)x_1 + u \qquad &= f_2(x_1, x_2, u) \\ y = x_1 \qquad\qquad\qquad &= g_1(x_1, x_2, u) \end{aligned}\right\} \Rightarrow \text{ Non-linear state-space representation}$$

2. Let $u_0 = 2,$ and solving for $X_e = \begin{pmatrix} x_{1e} \\ x_{1e} \end{pmatrix}$:

$$0 = f_1(x_{1e}, x_{2e}, u_0) = x_{2e}$$

$$0 = f_2(x_{1e}, x_{2e}, u_0) = -(1+x_{1e}^2)x_{1e} + 2$$

Using the MATLAB command roots, the roots of $x_{1e}^3 + x_{1e} - 2 = 0$
are $x_{1e} = 1, -0.5 \pm j1.323$.
As $x_1 = y$, the only physically meaningful equilibrium state is

$$X_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} x_{1e} \\ x_{2e} \end{pmatrix}$$

3.

$$F = \left.\begin{pmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{pmatrix}\right|_0 = \left.\begin{pmatrix} 0 & 1 \\ -1-3x_{1e} & 0 \end{pmatrix}\right|_0 = \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}$$

$$G = \left.\begin{pmatrix} \partial f_1/\partial u \\ \partial f_2/\partial u \end{pmatrix}\right|_0 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \qquad H = \left.\begin{pmatrix} \partial g/\partial x_1 & \partial g_1/\partial x_2 \end{pmatrix}\right|_0 = \begin{pmatrix} 1 & 0 \end{pmatrix} \qquad J = \partial g/\partial u = 0.$$

$$\left.\begin{aligned} \delta\dot{X} &= \begin{pmatrix} 0 & 1 \\ -4 & 0 \end{pmatrix}\delta X + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\delta u \\ \delta y &= \begin{pmatrix} 1 & 0 \end{pmatrix}\delta X \end{aligned}\right) \Rightarrow \text{ describes perturbations from } X_e = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad u_0 = 2$$

Solution of Problem 5:

الف) ماتريس انتقال حالت چنين است

$$\Phi(t) = e^{At} = I + At + \frac{1}{2!}A^2t^2 + \cdots$$

ولي A=0 ، بنابراين $A^3 = A^4 = \cdots = 0$. پس

$$\Phi(t) = e^{At} = I + At = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} t = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

ب)   معادله حالت در هر زمان $t \ge 0$ چنين است

$$x(t) = \Phi(t)x(0)$$

و از آنجا كه $x_1(0) = x_2(0) = 1$ به دست مي آوريم

$$x_1(t) = x_1(0) + tx_2(0) = 1 + t$$
$$x_2(t) = x_2(0) = 1$$

Solution of Problem 6:

$$y = x_1$$
$$\dot{y} = x_2$$
$$\ddot{y} = \dot{x}_2 = r(t) + (-1 - g_1)x_1(t) + (-2 - g_2)x_2(t)$$
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 - g_1 & -2 - g_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} r$$
$$\det(sI - A) = \begin{bmatrix} s & -1 \\ 1 + g_1 & s + 2 + g_2 \end{bmatrix} = (s+2)(s+3) \Rightarrow \begin{cases} g_1 = 5 \\ g_2 = 3 \end{cases}$$