Solution of Problem 1:

1. $A=\begin{pmatrix} 0 & 1 \\ 200 & -10 \end{pmatrix}$. The eigenvalues of satisfy $\det(\lambda I - A)=0$:

$$\det\begin{pmatrix} \lambda & -1 \\ -200 & \lambda+10 \end{pmatrix} = \lambda(\lambda+10)-200=0 \Rightarrow \qquad \lambda_1=-20 \quad \lambda_2=10$$

2. Because $\lambda_2 \rangle 0$, the system is not BIBS stable.

3.

$$\frac{Y(s)}{U(s)} = C(SI-A)^{-1}B+D \qquad (SI-A)^{-1}=\frac{1}{s^2+10s-200}\begin{pmatrix} s+10 & 1 \\ 200 & s \end{pmatrix}$$

$$\frac{Y(s)}{U(s)}=\begin{pmatrix} 1 & -10 \end{pmatrix}\begin{pmatrix} s+10 & 1 \\ 200 & s \end{pmatrix}\begin{pmatrix} 0 \\ 1 \end{pmatrix}\frac{1}{s^2+10s-200}=\frac{-10s+1}{(s+20)(s-10)}$$

4.

$$\frac{Y(s)}{U(s)}=\frac{-10\left(s-\frac{1}{10}\right)}{(s+20)(s-10)}$$

poles are at −20, +10

Because the input/output transfer function has a pole at $s=+10$, the system is not BIBO stable.

Suppose $C=\begin{pmatrix} -10 & 1 \end{pmatrix}$, then $\frac{Y(s)}{U(s)}=\begin{pmatrix} -10 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ s \end{pmatrix}\frac{1}{(s+20)(s-10)}=\frac{s-10}{(s-10)(s+20)}=\frac{1}{s+20}$

In this case $Y(s)/U(s)$ has a single pole at $s=-20$ and the system is BIBO stable. This example shows that BIBO stability does not imply BIBS stability.

Solution of Problem 2:

1. See the bellow m-file:

```
function [tr, ts, Mp, tp, yss] = find_resp_char(y,t);
% FIND_REP_CHAR time response characteristics of dynamic responses
% assumes last point of y vector represents steady-state value
yss = y(end);
% find Mp (percent) and tp
[Mp, tp_index] = max(y);
Mp = 100*(Mp - yss)/yss;
tp = t(tp_index);
% find tr; limit search to 0 < t < tp to find initial 10% to 90% rise in response
tr_range_indices = find( (y >= 0.1*yss) & (y <= 0.9*yss) & ( t <= tp));
tr = max( t(tr_range_indices) ) - min( t(tr_range_indices) );
% find ts by flipping t and y
yf = flipud(y);
tf = flipud(t);
ts_indices = find( (yf >= 1.01*yss) | (yf <= 0.99*yss) );
ts = tf ( min(ts_indices) );
```

2. for the signal supplied on the web page:

$t_r = 0.22 \text{ sec}$ $m_p = 35.1\%$

$t_s = 2.36 \text{ sec}$ $t_p = 0.55 \text{ sec}$

$y_{ss} = 10$

Solution of Problem 3:

1. The transfer function representation of the second order ODE $\ddot{y} + 2\zeta\omega_n\dot{y} + \omega_n{}^2 y = k\omega_n{}^2 u$ is derived as $Y(s)\{s^2 + 2\zeta\omega_n s + \omega_n{}^2\} = k\omega_n{}^2 U(s)$

$$H(s) = \frac{Y(s)}{U(s)} = \frac{k\omega_n{}^2}{s^2 + 2\zeta\omega_n s + \omega_n{}^2}$$

If the poles are in the left-half plane, the k represents the DC gain of the system or $H(0) = k$.

2. the roots of $s^2 + 2\zeta\omega_n s + \omega_n{}^2 = 0$ reveal that the poles are located at $s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$

For the <u>over damped</u> case, where ζ >1, the poles are real, negative, and distinct. As

$$s_1 s_2 = \left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}\right)\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}\right)$$
$$= \zeta^2\omega_n{}^2 - \zeta^2\omega_n{}^2 - \omega_n$$

We can write $H(s)$ as

$$H(s) = \frac{ks_1 s_2}{(s - s_1)(s - s_2)}.$$

The Laplace transform of the zero-state unit-step response is

$$Y(s) = H(s)U(s) = \frac{ks_1 s_2}{s(s - s_1)(s - s_2)}$$

To find $y(t)$, first expand $Y(s)$ as the sum of partial fractions:

$$Y(s) = k\left\{\frac{1}{s} + \frac{s_2}{s_1 - s_2}\frac{1}{s - s_1} - \frac{s_1}{s_1 - s_2}\frac{1}{s - s_2}\right\}$$

$$y(t) = k\left\{1 + \frac{s_2}{s_1 - s_2}e^{s_1 t} - \frac{s_1}{s_1 - s_2}e^{s_2 t}\right\}$$

3. when $\zeta = 1$, the poles are located at

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1} = -\omega_n$$

Hence the poles are real, negative and identical.
In this case

$$Y(s) = H(s)\frac{1}{s} = \frac{k\omega_n{}^2}{s(s + \omega_n)^2}$$

Expand $Y(s)$ as

$$Y(s) = \frac{c_1}{s} + \frac{c_2}{s + \omega_n} + \frac{c_3}{(s + \omega_n)^2} \Rightarrow \qquad Y(s) = k\left\{\frac{1}{s} - \frac{1}{s + \omega_n} - \frac{\omega_n}{(s + \omega_n)^2}\right\}$$

$$y(t) = k\{1 - e^{-\omega_n t} - t\omega_n e^{-\omega_n t}\}$$

4. When $0 \le \zeta < 1$ the poles are complex conjugates with a negative real part. For this case express $H(s)$ as

$$H(s) = \frac{k\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{k\omega_n^2}{\left(s + \zeta\omega_n\right)^2 + \omega_n^2 - \zeta^2\omega_n^2} = \frac{k\omega_n^2}{\left(s+\sigma\right)^2 + \omega_d^2}$$

So that we can expand $Y(s)$ as

$$Y(s) = \frac{k\omega_n^2}{s\left\{(s+\sigma)^2 + \omega_d^2\right\}} = k\left\{\frac{1}{s} - \frac{s + 2\sigma}{(s+\sigma)^2 + \omega_d^2}\right\}$$

$$= k\left\{\frac{1}{s} - \frac{s+\sigma}{(s+\sigma)^2 + \omega_d^2} - \frac{\sigma}{\omega_d}\frac{\omega_d}{(s+\sigma)^2 + \omega_d^2}\right\}$$

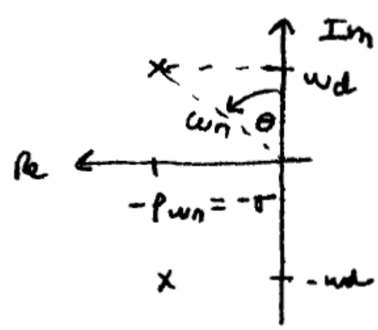
From the elementary transform pairs we obtain

$$y(t) = k\left\{1 - e^{-\sigma t}\cos(\omega_d t) - \frac{\sigma}{\omega_d}e^{-\sigma t}\sin(\omega_d t)\right\}$$

$$= k\left\{1 - e^{-\sigma t}\left(\cos(\omega_d t) + \frac{\sigma}{\omega_d}\sin(\omega_d t)\right)\right\}$$

5. Once again consider the under damped case where $0 \le \zeta < 1$. (Note that the complex conjugated poles are located ω_n away from the origin.

Consider the pole $s_1 = -\zeta\omega_n + j\omega_d$ in the second quadrant:



Note that $\sin\theta = \zeta$ and so $\zeta = \sin^{-1}\theta$

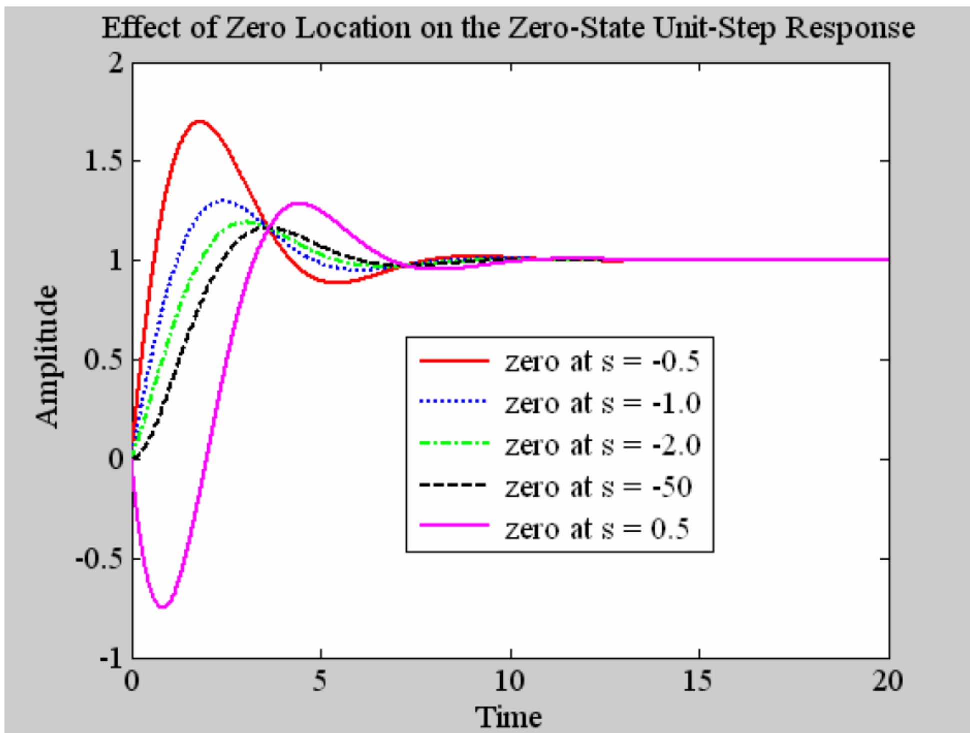
6. $\mathrm{Re}\{s_{1,2}\} = \sigma = -\zeta\omega_n$. the natural frequency ω_n is always greater than or equal to zero. And so if $\zeta < 0$ and $\omega_n > 0$, the pole(s) will be located in the right-half plane. As a result, the system is unstable and the DC gain is undefined.

Solution of Problem 4:

1. This problem considers the effect of a finite zero on the unit-step response of a second-order system:

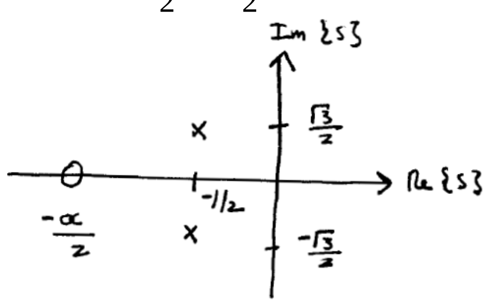
$$\frac{Y(s)}{U(s)}=\frac{\frac{s}{\alpha\xi\omega_n}+1}{\left(\frac{s}{\omega_n}\right)^2+2\xi\left(\frac{s}{\omega_n}\right)+1}$$

With $\zeta=0.5$ and $\omega_n=1$, the unit-step responses are shown for $\alpha=-1,1,2,4$ *and* 100. The corresponding m-file is given also.



```
% clear the workspace and close all figures
clear
close all

% define the denominator of the system transfer function
zeta    = 0.5;
omega_n = 1;
den = [1/omega_n^2, 2*zeta*omega_n, 1];

% define the numerator and system for each value of alpha
alpha =   1; num = [1/(alpha*zeta*omega_n), 1]; sys1 = tf(num, den);
alpha =   2; num = [1/(alpha*zeta*omega_n), 1]; sys2 = tf(num, den);
alpha =   4; num = [1/(alpha*zeta*omega_n), 1]; sys3 = tf(num, den);
alpha = 100; num = [1/(alpha*zeta*omega_n), 1]; sys4 = tf(num, den);
alpha =  -1; num = [1/(alpha*zeta*omega_n), 1]; sys5 = tf(num, den);

% Compute the zero-state unit-step response for each system, show
% time-response characteristics
t = linspace(0,20,10000)';
y(:,1) = step(sys1, t); [tr1, ts1, Mp1, tp1, yss] = find_resp_char(y(:,1),
y(:,2) = step(sys2, t); [tr2, ts2, Mp2, tp2, yss] = find_resp_char(y(:,2),
y(:,3) = step(sys3, t); [tr3, ts3, Mp3, tp3, yss] = find_resp_char(y(:,3),
y(:,4) = step(sys4, t); [tr4, ts4, Mp4, tp4, yss] = find_resp_char(y(:,4),
y(:,5) = step(sys5, t);

% plot the step responses
figure(1)
handle = plot(t, y(:,1), '-r', t, y(:,2), ':b', t, y(:,3), '-.g', ...
              t, y(:,4), '--k', t, y(:,5), '-m');
set(handle, 'LineWidth', 2, 'MarkerSize', 10);
set(gca, 'FontSize', 14, 'FontName', 'times new roman');
legend('\fontsize{14}\fontname{times new roman} zero at s = -0.5', ...
       '\fontsize{14}\fontname{times new roman} zero at s = -1.0', ...
       '\fontsize{14}\fontname{times new roman} zero at s = -2.0', ...
       '\fontsize{14}\fontname{times new roman} zero at s = -50',...
       '\fontsize{14}\fontname{times new roman} zero at s = 0.5');
title('Effect of Zero Location on the Zero-State Unit-Step Response', ...
      'FontSize', 14, 'FontName', 'Times New Roman')
ylabel('Amplitude', 'FontSize', 14, 'FontName', 'Times New Roman')
xlabel('Time', 'FontSize', 14, 'FontName', 'Times New Roman')
```

2. The poles are located at $s = -\zeta\omega_n \pm j\omega_d = -\frac{1}{2} \pm j\frac{\sqrt{3}}{2}$. The finite zero is located at $s = -\alpha\zeta\omega_n = -\frac{\alpha}{2}$



α	Zero location
-1	$1/2$
1	$-1/2$
2	-1
4	-2
100	-50

Using the up m-file, and the function find-resp-char.m (from problem2) the percent overshoot, the time-to-peak, rise-time, and settling time are determined for the cases where the zero is located in left-half plane.

Zero locations	$t_r\,[s]$	$t_s\,[s]$	$m_p\,\%$	$t_p\,[s]$
$-1/2$	0.476	10.3	69.9	1.81
-1	0.938	7.92	29.8	2.42
-2	1.37	8.28	19.1	3.02
-50	1.63	8.76	16.3	3.61

As the zero moves towards the $j\omega$ axis, the overshoot increases significantly, while the rise-time decreases. Also note that the settling tme also increases significantly when the zero approaches $s = 0$.

3. As seen in the previous plot, when the zero is located in the right-half plane, the unit-step response initially moves *away* from the steady-state value of the response, rather than towards it.