LINEAR CONTROL SYSTEMS

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Lecture 10

Time domain analysis of control systems

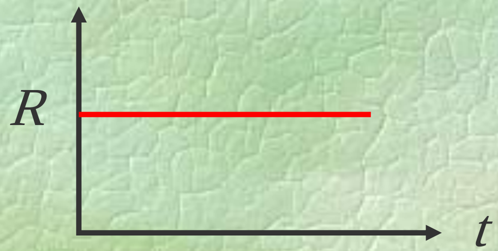
Topics to be covered include:

Time domain analysis.

- Test signals.
- Steady state error and error coefficients
- Error series
- Introducing some performance criteria (ISE, ITSE, IAE and ITAE).

Some test signals

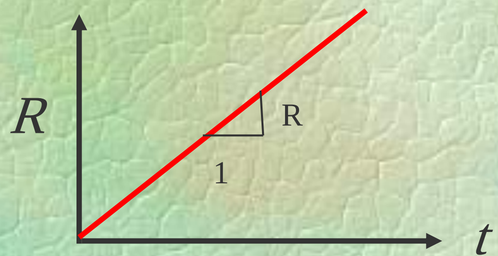
چند سیگنال تست



Step input

ورودی پله

$$r(t) = Ru(t) \qquad R(s) = \frac{R}{s}$$



Velocity input

ورودی شیب

$$r(t) = Rtu(t) \qquad R(s) = \frac{R}{s^2}$$

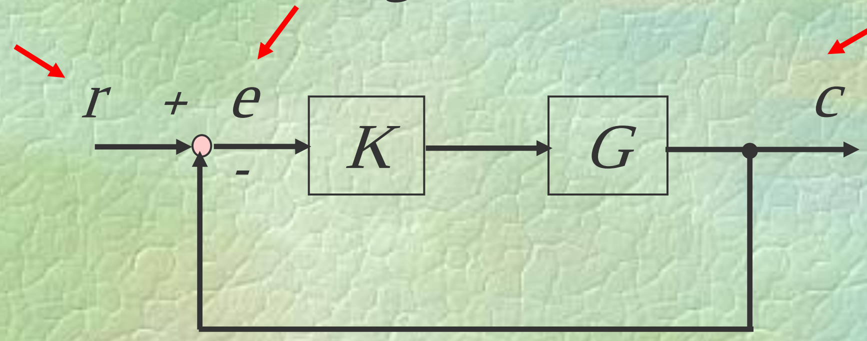
Acceleration input

ورودی شتاب

$$r(t) = \frac{Rt^2}{2} u(t) \qquad R(s) = \frac{R}{s^3}$$

Error in control systems

خطا در سیستمهای کنترل



$e(t)=r(t)-c(t)$

$E(s)=R(s)-C(s)$

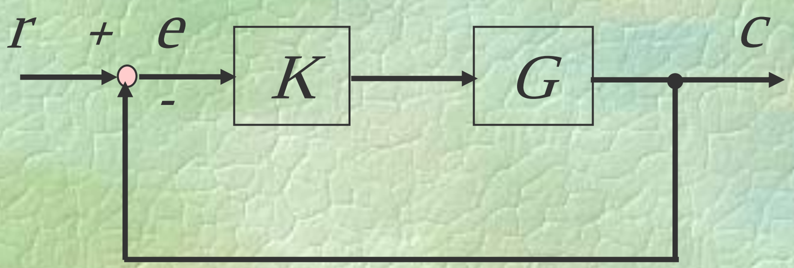
Loop transfer function

$$G(s)K(s) = \frac{k(1+\tau_1 s)(1+\tau_2 s).......(1+\tau_m s)}{s^j(1+\tau_{d1} s)(1+\tau_{d2} s).......(1+\tau_{dn} s)} e^{-T_d s}$$

$j=0$	type zero system	سیستم نوع صفر
$j=1$	type one system	سیستم نوع یک
$j=2$	type two system	سیستم نوع دو
.	.	.
.	.	.

Error in control systems

خطا در سیستمهای کنترل



$$T(s)=\frac{G(s)K(s)}{1+G(s)K(s)}$$

$$E(s)=R(s)-T(s)R(s)$$

$$E(s)=\frac{1}{1+G(s)K(s)}R(s)$$

If the system is stable:

(very important)

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\boxed{\lim_{s\to 0}s\big(R(s)-T(s)R(s)\big)} \quad (1)$$

$$e_{ss}=\lim_{t\to\infty}e(t)=\lim_{s\to 0}sE(s)=\boxed{\lim_{s\to 0}\frac{s}{1+G(s)K(s)}R(s)} \quad (2)$$

Error in control systems for step input

خطا در سیستمهای کنترل برا ی ورودی پله

1

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\big(R(s)-T(s)R(s)\big)$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{s}{1+G(s)K(s)}R(s)$$

2

$$R(s)=\frac{R}{s}$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0} s\left(\frac{R}{s}-T(s)\frac{R}{s}\right)=R\left(1-\lim_{s\to 0}T(s)\right)$$

$$e_{ss}=\lim_{s\to 0} sE(s)=\lim_{s\to 0}\frac{s}{1+G(s)K(s)}\frac{R}{s}=\frac{R}{1+\lim_{s\to 0}G(s)K(s)}=\frac{R}{1+K_p}$$

$$K_p=\lim_{s\to 0}G(s)K(s)$$

Position constant

Error in control systems for velocity input

خطا در سیستمهای کنترل برا ی ورودی شیب

1

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\big(R(s) - T(s)R(s)\big)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G(s)K(s)} R(s)$$

2

$$R(s) = \frac{R}{s^2}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{R}{s^2} - T(s)\frac{R}{s^2}\right) = R\lim_{s\to 0}\left(\frac{1-T(s)}{s}\right)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G(s)K(s)} \frac{R}{s^2} = \frac{R}{0+\lim_{s\to 0} sG(s)K(s)} = \frac{R}{K_v}$$

$$K_v = \lim_{s\to 0} sG(s)K(s)$$

Velocity constant

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Error in control systems for parabolic input

خطا در سیستمهای کنترل برا ی ورودی پارابولیک

1

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\big(R(s) - T(s)R(s)\big)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G(s)K(s)} R(s)$$

2

$$R(s) = \frac{R}{s^3}$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} s\left(\frac{R}{s^3} - T(s)\frac{R}{s^3}\right) = R\lim_{s\to 0}\left(\frac{1-T(s)}{s^2}\right)$$

$$e_{ss} = \lim_{s\to 0} sE(s) = \lim_{s\to 0} \frac{s}{1+G(s)K(s)} \frac{R}{s^3} = \frac{R}{0+\lim_{s\to 0} s^2 G(s)K(s)} = \frac{R}{K_a}$$

$$K_a = \lim_{s\to 0} s^2 G(s)K(s)$$

Acceleration constant

Error in control systems

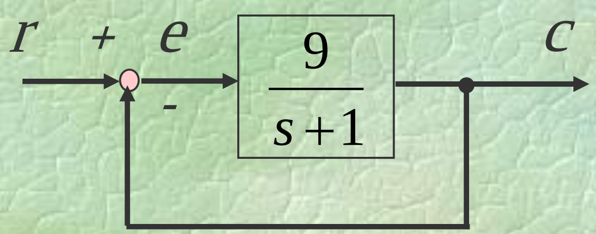
خطا در سیستمهای کنترل

$$G(s)K(s)=\frac{k(1+\tau_1 s)(1+\tau_2 s).......(1+\tau_m s)}{s^j(1+\tau_{d1}s)(1+\tau_{d2}s).......(1+\tau_{dn}s)}e^{-T_d s}$$

Type	K_p	K_v	K_a	e_{ss} (position)	e_{ss} (velocity)	e_{ss} (acceleration)
0	k	0	0	$\frac{R}{1+k}$	∞	∞
1	∞	k	0	0	$\frac{R}{k}$	∞
2	∞	∞	k	0	0	$\frac{R}{k}$
3	∞	∞	∞	0	0	0

Example 1: Find the different errors in following system.

مثال ۱: خطاهای مختلف را برای سیستم زیر تعیین کنید.



$$G(s)K(s)=\frac{9}{s+1} \qquad T(s)=\frac{9}{s+10}$$

System is stable so we continue

Error for unit step input

$$K_p=\lim_{s\to 0}G(s)K(s)=9 \qquad e_{ss}=\frac{1}{1+9}=0.1$$

or $$e_{ss}=1\left(1-\lim_{s\to 0}T(s)\right)=1-0.9=0.1$$

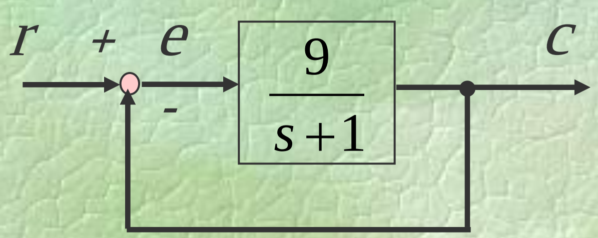
Errors for unit velocity and unit parabolic input

$$K_v=K_a=0 \qquad e_{ss}=\infty$$

or $$e_{ss}=1\lim_{s\to 0}\left(\frac{1-T(s)}{s}\right)=\infty$$

Example 1: Step response

مثال ۱: پاسخ پله

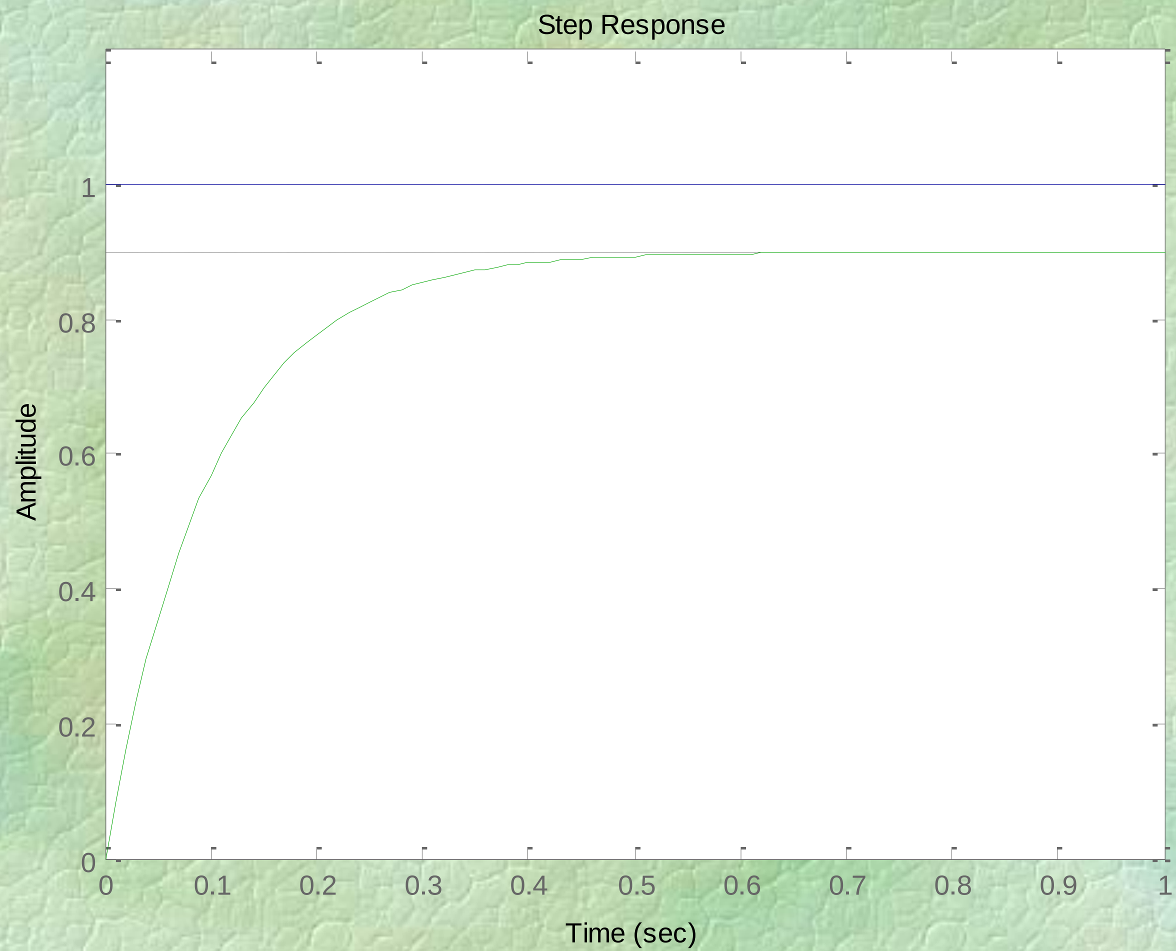


$$T(s)=\frac{9}{s+10}$$

MATLAB

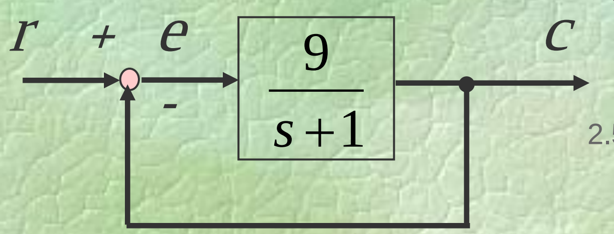
step(9,[1 10])

hold on;step(1,1)



Example 1: Velocity response

مثال ١: پاسخ شیب یا سرعت

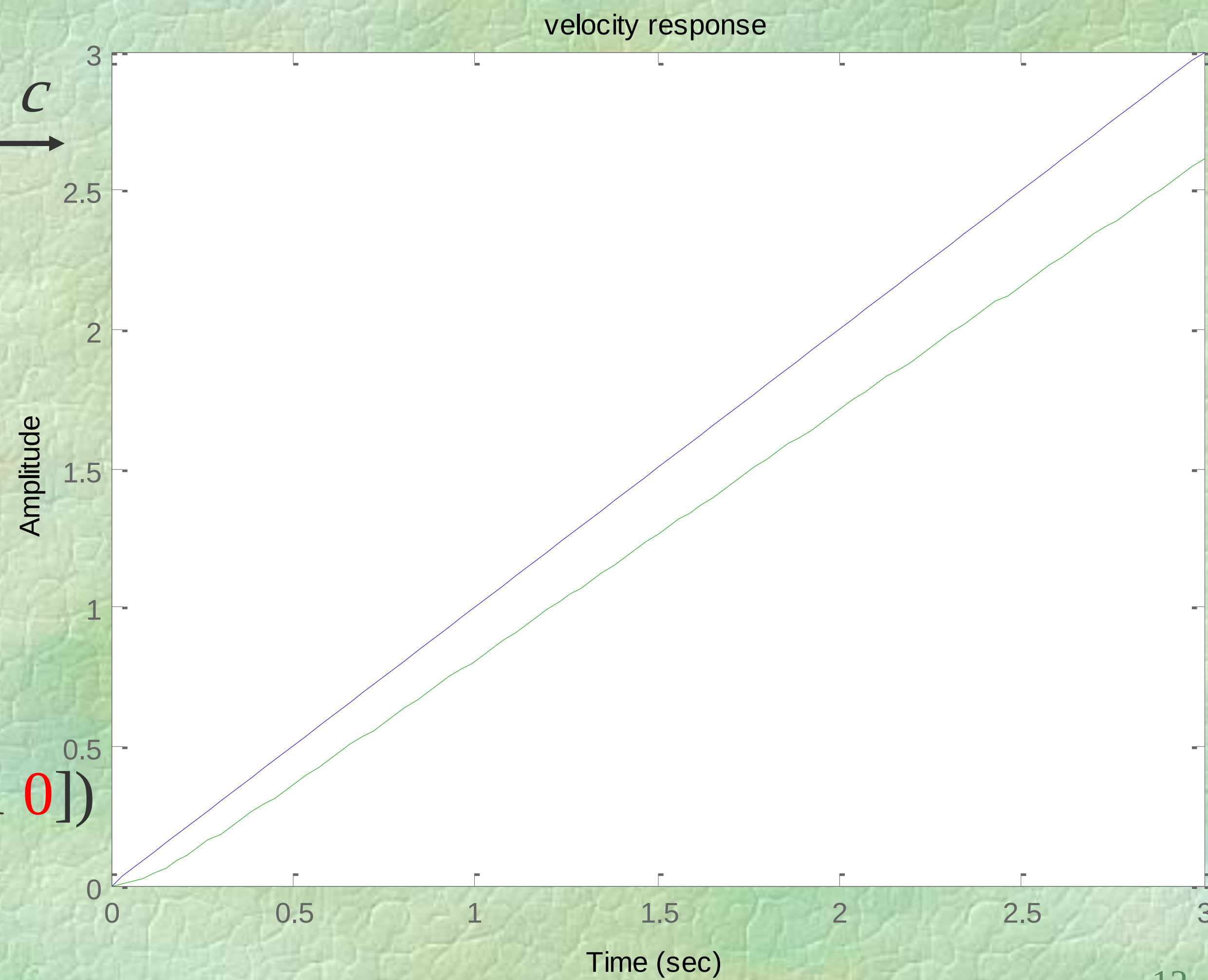


$$T(s)=\frac{9}{s+10}$$

MATLAB

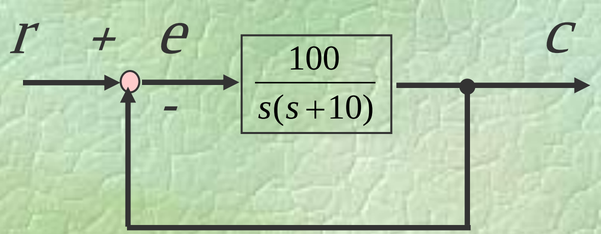
step(9,[1 10 0])

hold on;step(1,[1 0])



Example 2: Find the different errors in following system

مثال ۲: خطاهای مختلف را برای سیستم زیر تعیین کنید



$$G(s)K(s)=\frac{100}{s(s+10)} \qquad T(s)=\frac{100}{s^2+10s+100}$$

System is stable so we continue

Error for step input

$$K_p=\lim_{s\to 0} G(s)K(s)=\infty \qquad e_{ss}=\frac{1}{1+\infty}=0 \qquad \text{or} \qquad e_{ss}=1\left(1-\lim_{s\to 0} T(s)\right)=1-1=0$$

Error for velocity input

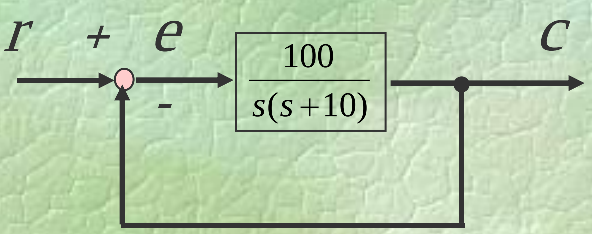
$$K_v=\lim_{s\to 0} sG(s)K(s)=10 \qquad e_{ss}=\frac{1}{10}=0.1 \qquad \text{or} \qquad e_{ss}=1\lim_{s\to 0}\left(\frac{1-T(s)}{s}\right)=0.1$$

Error for parabolic input

$$K_a=0 \qquad e_{ss}=\infty \qquad \text{or} \qquad e_{ss}=1\lim_{s\to 0}\left(\frac{1-T(s)}{s^2}\right)=\infty$$

Example 2: Step response

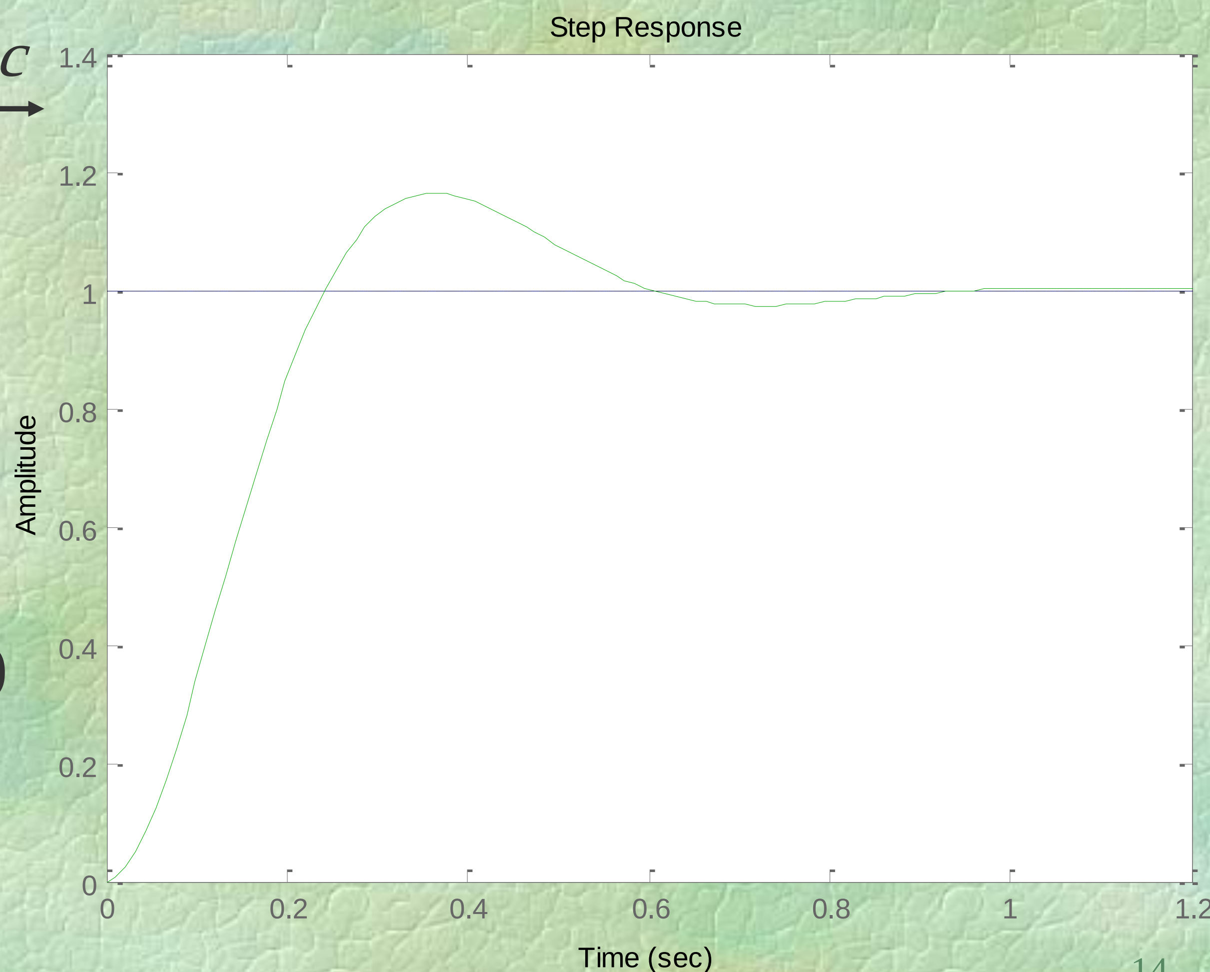
مثال ۲: پاسخ پله



$$T(s)=\frac{100}{s^2+10s+100}$$

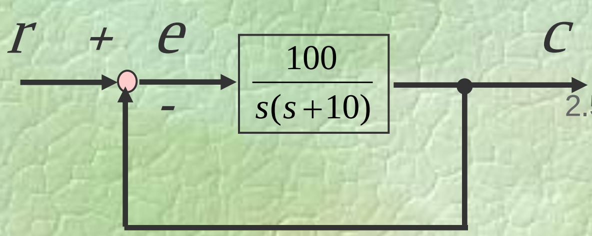
MATLAB

```
step(100,[1 10 100])
hold on;step(1,1)
```



Example 2: Velocity response

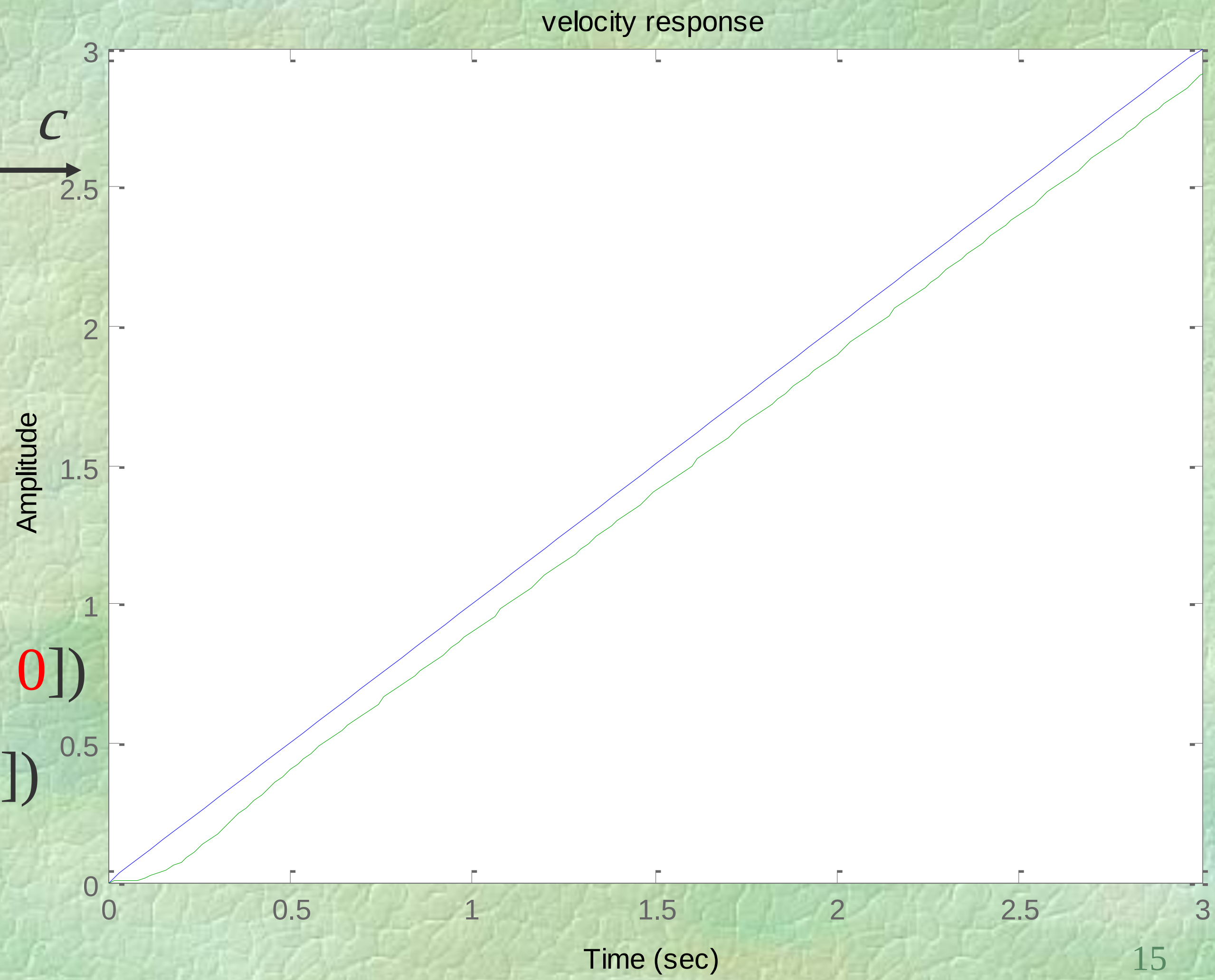
مثال ۲: پاسخ شیب یا سرعت



$$T(s)=\frac{100}{s^2+10s+100}$$

step(100,[1 10 100 0])

hold on;step(1,[1 0])



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Example 3: The closed loop transfer function of a system is given. Determine *a* such that the system error to step input is zero.

$$T(s)=\frac{a}{s^3+12s^2+6s+23}$$

مثال ۳: برای سیستم حلقه بسته داده شده *a* را چنان تنظیم کنید که خطای سیستم به ورودی پله صفر گردد

First of all check the stability:

s^3	1	6
s^2	12	23
s^1	$\frac{49}{12}$	0
s^0	23	

Clearly it is stable

$$e_{ss}=R\left(1-\lim_{s\to 0}T(s)\right)=0$$

$$\lim_{s\to 0}T(s)=1$$

$$\frac{a}{23}=1 \quad \Rightarrow \quad a=23$$

Example 4: The closed loop transfer function of a system is given. Determine *a* and *b* such that the system error to velocity input is zero.

مثال ۴: برای سیستم حلقه بسته داده شده a و b را چنان تنظیم کنید که خطای سیستم به ورودی سرعت صفر گردد

$$T(s)=\frac{as+b}{s^3+12s^2+6s+23}$$

First of all check the stability: It is stable by example 3

$$e_{ss}=R\lim_{s\to 0}\left(\frac{1-T(s)}{s}\right)=R\lim_{s\to 0}\left(\frac{1-\frac{as+b}{s^3+12s^2+6s+23}}{s}\right)$$

$$e_{ss}=R\lim_{s\to 0}\left(\frac{s^3+12s^2+(6-a)s+(23-b)}{s\left(s^3+12s^2+6s+23\right)}\right)=0$$

$$a=6$$

$$b=23$$

Example 5: The closed loop transfer function of a system is given. Find the system error to unit step.

$$T(s)=\frac{as+b}{s^3+12s^2-6s+23}$$

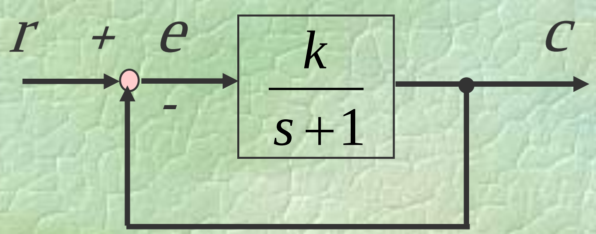
مثال ۵: برای سیستم حلقه بسته داده شده خطای سیستم به ورودی پله واحد را بیابید

First of all check the stability:

It is unstable so the error is infinity

Example 6: Find the different errors in following system

$k \geq 0$ مثال ۶: خطاهای مختلف را برای سیستم زیر تعیین کنید



$$G(s)K(s) = \frac{k}{s+1} \qquad T(s) = \frac{k}{s+1+k}$$

System is stable so we continue

Error for step input

$$K_p = \lim_{s\to 0} G(s)K(s) = k \qquad e_{ss} = \frac{1}{1+k}$$

Errors for velocity and parabolic input $K_v = K_a = 0$ $e_{ss} = \infty$

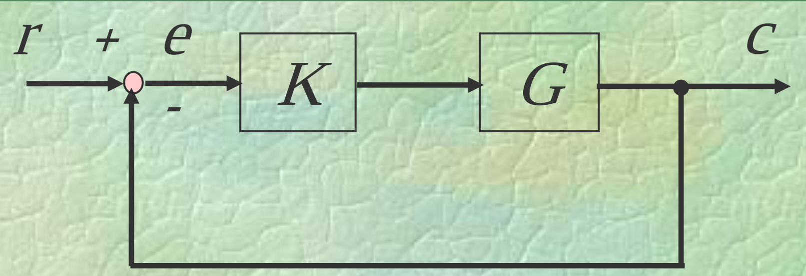
Note that the above method doesn’t say anything about how the errors go to infinity

توجه کنید که این روش راجع به چگونگی میل خطا به بینهایت صحبت نمی کند.

Error series can explain the matter.

Error series

سری خطا



$$E(s) = W_e(s)R(s) \qquad e(t) = \int_0^t w_e(\tau) r(t-\tau) d\tau$$

Expand $r(t-\tau)$ around t :

$$r(t-\tau) = r(t) - r'(t)\tau + \frac{r''(t)}{2!}\tau^2 - \frac{r'''(t)}{3!}\tau^3 + - ....$$

$$e(t) = r(t)\int_0^t w_e(\tau)d\tau - r'(t)\int_0^t \tau\, w_e(\tau)d\tau + r''(t)\int_0^t \frac{\tau^2}{2!} w_e(\tau)d\tau - ....$$

Error series

سری خطا

$$e(t) = r(t)\int_0^t w_e(\tau)d\tau - r'(t)\int_0^t \tau\, w_e(\tau)d\tau + \frac{r''(t)}{2!}\int_0^t \tau^2\, w_e(\tau)d\tau - ....$$

Now consider steady value for *r*

$$e_s(t) = r_s(t)\underbrace{\int_0^\infty w_e(\tau)d\tau}_{C_0} - r_s'(t)\underbrace{\int_0^\infty \tau\, w_e(\tau)d\tau}_{C_1} + \frac{r_s''(t)}{2!}\underbrace{\int_0^\infty \tau^2\, w_e(\tau)d\tau}_{C_2} - ....$$

$$e_s(t) = C_0 r_s(t) + C_1 r_s'(t) + \frac{C_2}{2!} r_s''(t) + \frac{C_3}{3!} r_s'''(t) + ...$$

Error series coefficients ضرائب سری خطا

$$e_s(t) = C_0 r_s(t) + C_1 r_s'(t) + \frac{C_2}{2!} r_s''(t) + \frac{C_3}{3!} r_s'''(t) + ...$$

$$C_0 = \int_0^\infty w_e(\tau) d\tau \qquad C_1 = -\int_0^\infty \tau\, w_e(\tau) d\tau \qquad C_2 = \int_0^\infty \tau^2\, w_e(\tau) d\tau \qquad ....$$

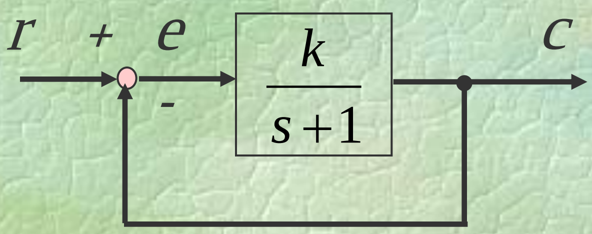
Calculation of coefficients محاسبه ضرائب

$$W_e(s) = \int_0^\infty w_e(\tau) e^{-\tau s} d\tau$$

$$C_0 = \lim_{s\to 0} W_e(s) \qquad C_1 = \lim_{s\to 0} \frac{dW_e(s)}{ds} \qquad .... \qquad C_n = \lim_{s\to 0} \frac{d^n W_e(s)}{ds^n} \qquad ....$$

Example 7: Determine the error series coefficients for system in example 1

مثال۷: ضرائب سری خطا را برای سیستم مثال ۱ بیابید



$$E(s)=\frac{s+1}{s+1+k}R(s)$$

$W_e(s)$

$$C_0=\lim_{s\to 0}W_e(s)=\frac{1}{1+k}$$

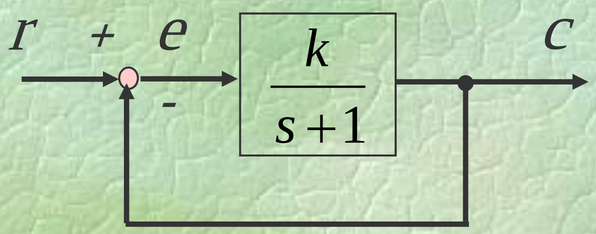
$$C_1=\lim_{s\to 0}\frac{dW_e(s)}{ds}=\frac{k}{(1+k)^2}$$

$$C_2=\lim_{s\to 0}\frac{d^2W_e(s)}{ds^2}=\frac{-2k}{(1+k)^3}$$

....

Example 8: Determine the error series for step and velocity inputs in example 1.

مثال۸: سری خطا را برای ورودیهای پله و شیب در مثال ۱بیابید.



$$C_0 = \frac{1}{1+k}, C_1 = \frac{k}{(1+k)^2}, C_2 = \frac{-2k}{(1+k)^2}, ...$$

Step: $r_s(t) = 1, r_s'(t) = r_s''(t) = ... = 0$

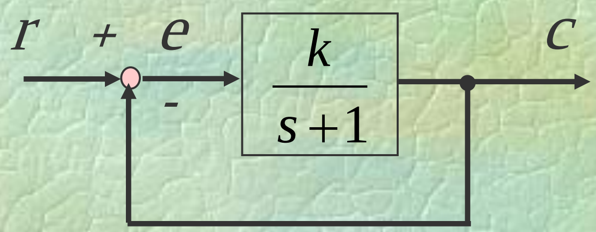
$$e_s(t) = C_0 r_s(t) + C_1 r_s'(t) + \frac{C_2}{2!} r_s''(t) + \frac{C_3}{3!} r_s'''(t) + ... = \frac{1}{1+k}$$

Ramp: $r_s(t) = t, r_s'(t) = 1, r_s''(t) = r_s'''(t) = ... = 0$

$$e_s(t) = C_0 r_s(t) + C_1 r_s'(t) + \frac{C_2}{2!} r_s''(t) + \frac{C_3}{3!} r_s'''(t) + ... = \frac{t}{1+k} + \frac{k}{(1+k)^2}$$

Example 9: Compare the result of example 1 and 7

مثال۹: نتایج مثالهای ۱ و ۷ را مقایسه کنید



	Example 1	Example 7
Step:	$e_{ss}=\frac{1}{1+k}$	$e_s(t)=\frac{1}{1+k}$
Ramp:	$e_{ss}=\infty$	$e_s(t)=\frac{t}{1+k}+\frac{k}{(1+k)^2}$

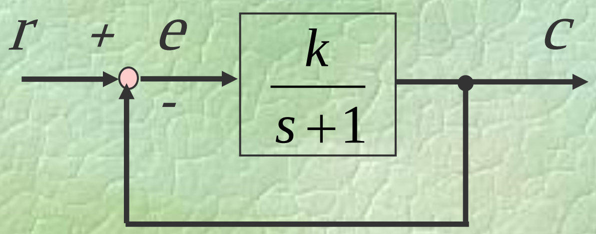
They have similar result but error series show how the error go to infinity

مشخص است که نتایج یکسان است ولی سری خطا نحوه میل به بینهایت را نیز نشان می دهد.

Example 10: Determine the error for following input

k=100

مثال ۱۰: خطا را برای ورودی داده شده بیابید



$$r(t) = \sin \omega_0 t \qquad \omega_0 = 2$$

This problem can **just** be solve with error series!

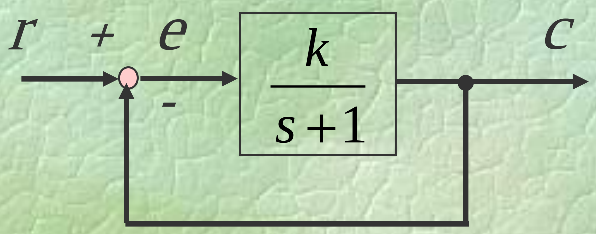
By example 8 we have $C_0 = \dfrac{1}{1+k}, C_1 = \dfrac{k}{(1+k)^2}, C_2 = \dfrac{-2k}{(1+k)^3}, \ldots\ldots$

$$r_s(t) = \sin \omega_0 t, r_s'(t) = \omega_0 \cos \omega_0 t,\ r_s''(t) = -\omega_0^{\ 2} \sin \omega_0 t, r_s'''(t) = -\omega_0^{\ 3} \cos \omega_0 t \ldots\ldots$$

$$e_s(t) = (C_0 - \frac{C_2}{2!}\omega_0^{\ 2} + \ldots)\sin \omega_0 t + (C_1\omega_0 - \frac{C_3}{3!}\omega_0^{\ 3} + \ldots)\cos \omega_0 t$$

Example 10: Determine the error for following input (continue)

مثال ۱۰: خطا را برای ورودی داده شده بیابید(ادامه)



$$r(t) = \sin\omega_0 t \qquad \omega_0 = 2 \qquad k = 100$$

$$e_s(t) = (C_0 - \frac{C_2}{1.2}\omega_0{}^2 + \ldots)\sin\omega_0 t + (C_1\omega_0 - \frac{C_3}{1.2.3}\omega_0{}^3 + \ldots)\cos\omega_0 t$$

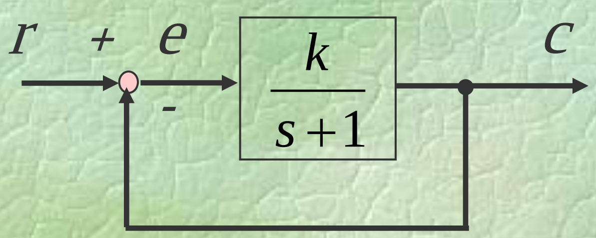
$$C_0 = \frac{1}{101} = 0.0099, C_1 = \frac{100}{(101)^2} = 0.0098, C_2 = \frac{-200}{(101)^3} = -0.000194, C_3 = \frac{600}{(101)^4} = 5.766\times10^{-6}$$

$$e_s(t) \cong (0.0099 - \frac{-0.000194}{1.2}\times4)\sin 2t + (0.0098\times2 - \frac{5.766\times10^{-6}}{1.2.3}\times8)\cos 2t$$

$$e_s(t) = 0.010288\sin 2t + 0.019592\cos 2t = 0.02213\sin(2t + 62.3^\circ)$$

Example 11: Determine the exact value of error in example 8.

مثال ۱۱: مقدار دقیق خطا را برای مثال ۸ بیابید.



$$r(t)=\sin\omega_0 t \quad \omega_0=2 \quad k=100$$

$$E(s)=\frac{s+1}{s+1+k}R(s)=\frac{s+1}{s+101}\frac{2}{s^2+4}$$

$$E(s)=\frac{\frac{-200}{101^2+4}}{s+101}+\frac{\frac{-4j+2}{(-2j+101)(-4j)}}{s+2j}+\frac{\frac{4j+2}{(2j+101)(4j)}}{s-2j}$$

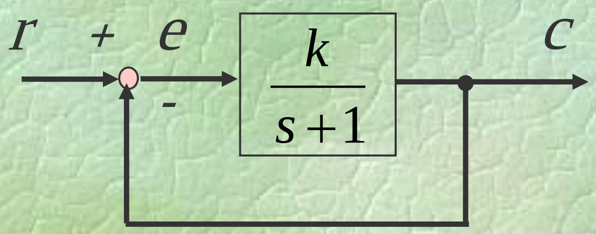
$$e(t)=-0.0196e^{-101t}+(0.0098+0.0047j)e^{-2jt}+(0.0098-0.0047j)e^{2jt}$$

$$e_s(t)=(0.0098+0.0047j)e^{-2jt}+(0.0098-0.0047j)e^{2jt}$$

$$e_s(t)=0.0094\sin 2t+0.0196\cos 2t=0.0217\sin(2t+64.4^\circ)$$

Example 12: Determine the response example 5.

مثال۱۲: پاسخ سیستم را برای مثال ۵ بیابید.

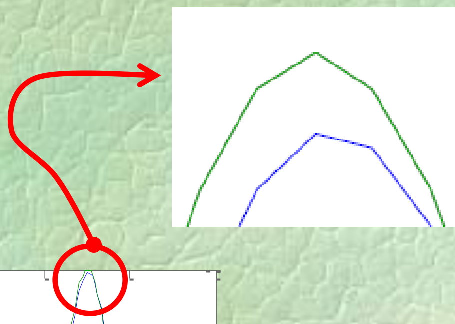


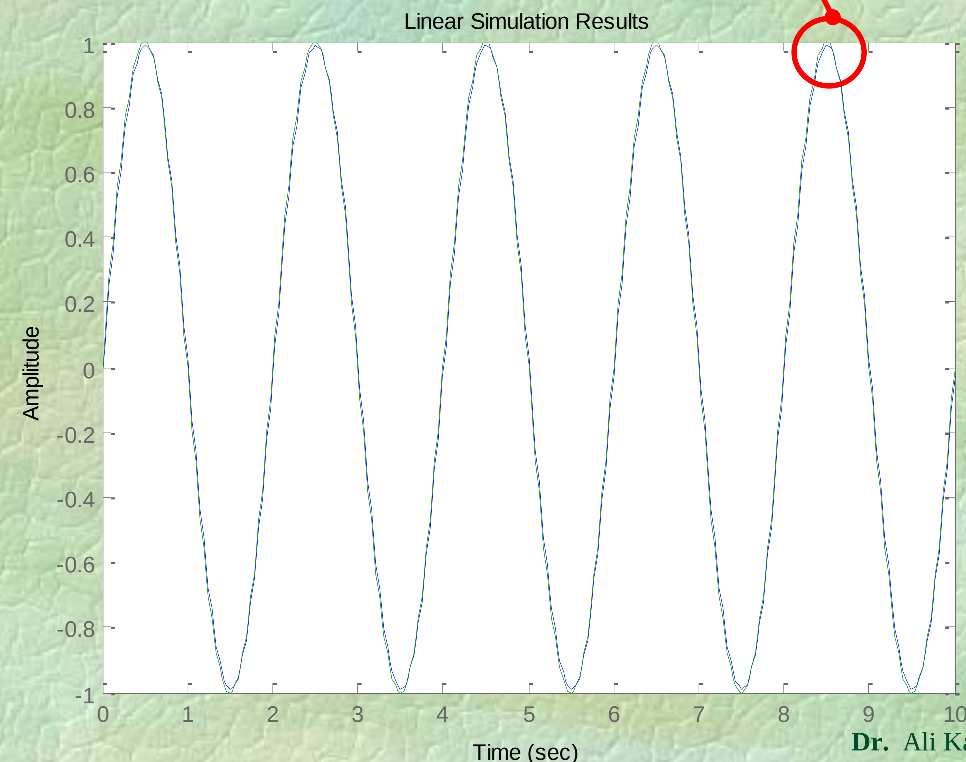
$$r(t) = \sin\omega_0 t \qquad \omega_0 = 2 \qquad k = 100$$

$$\frac{c(s)}{r(s)} = \frac{100}{s+101}$$

MATLAB

```
[u,t]=gensig('sin',2);
T1=tf(100,[1 101])
lsim(T1,u,t);
hold on;
T2=tf(1,1)
lsim(T2,u,t);
```





Performance indices

شاخص های عملکردی

$$ISE = \int_0^T e^2(t)dt$$

Integral of the **S**quare of the **E**rror

$$IAE = \int_0^T |e(t)|dt$$

Integral of the **A**bsolute magnitude of the **E**rror

Note that these are positive values and lower value is better

$$ITSE = \int_0^T te^2(t)dt$$

Integral of **T**ime multiplied by the **S**quared **E**rror

$$ITAE = \int_0^T t|e(t)|dt$$

Integral of **T**ime multiplied by the **A**bsolute **E**rror

This performance is used when the initial value of error is not very important

Exercises

1- Find the error of following system to step input.

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = [1 \quad 2 \quad 0] x$$

2- Find the error of the following systems to step input.

$$a)\ M(s) = \frac{2000}{s^3 + 15s^2 + 50s + 2000} \qquad b)\ M(s) = \frac{200}{s^3 + 15s^2 + 50s + 200} \qquad c)\ M(s) = \frac{500}{s^3 + 15s^2 + 50s + 600}$$

3- Find the error of the following systems to velocity input.

$$a)\ M(s) = \frac{50s + 2000}{s^3 + 15s^2 + 50s + 2000} \qquad b)\ M(s) = \frac{50s + 200}{s^3 + 15s^2 + 50s + 200}$$

Exercises(continue)

4- Repeat example 7 for parabolic input.

5- Find the error of following system to step input(Final 1391).