LINEAR CONTROL SYSTEMS

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Lecture 11

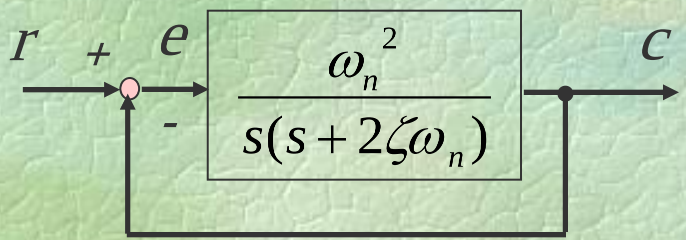
Time domain analysis of control systems

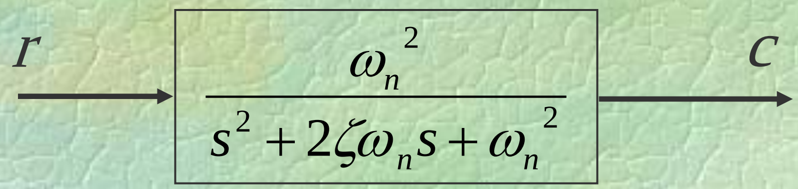
***Topics to be covered include*:**

- Time domain analysis.
  - Introducing a prototype second order system.
- Transient response of a prototype second order system
  - Damping ratio and damping factor.
  - Natural undamped frequency and damped frequency.
  - Percent overshoot.
  - Delay time, rise time and settling time.

Introducing a prototype second order system.

معرفی یک سیستم نمونه درجه ۲





$$\text{Poles are: } -\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} \quad \text{if } 0\le\zeta\le 1$$

$$C(s)=\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s) \quad \xrightarrow{\text{Step response}} \quad C(s)=\frac{\omega_n^2}{s(s^2+2\zeta\omega_n s+\omega_n^2)}$$

$$C(s)=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

Introducing a prototype second order system.

معرفی یک سیستم نمونه درجه ۲

$$C(s)=\frac{1}{s}-\frac{(s+\zeta\omega_n)}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}-\frac{\zeta\omega_n}{(s+\zeta\omega_n)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}$$

$$c(t)=u(t)\left(1-e^{-\zeta\omega_n t}\left(\cos(\omega_n\sqrt{1-\zeta^2}\,t)+\frac{\zeta}{\sqrt{1-\zeta^2}}\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right)\right)$$

$$c(t)=u(t)\left(1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sqrt{1-\zeta^2}\cos(\omega_n\sqrt{1-\zeta^2}\,t)+\zeta\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right)\right)$$

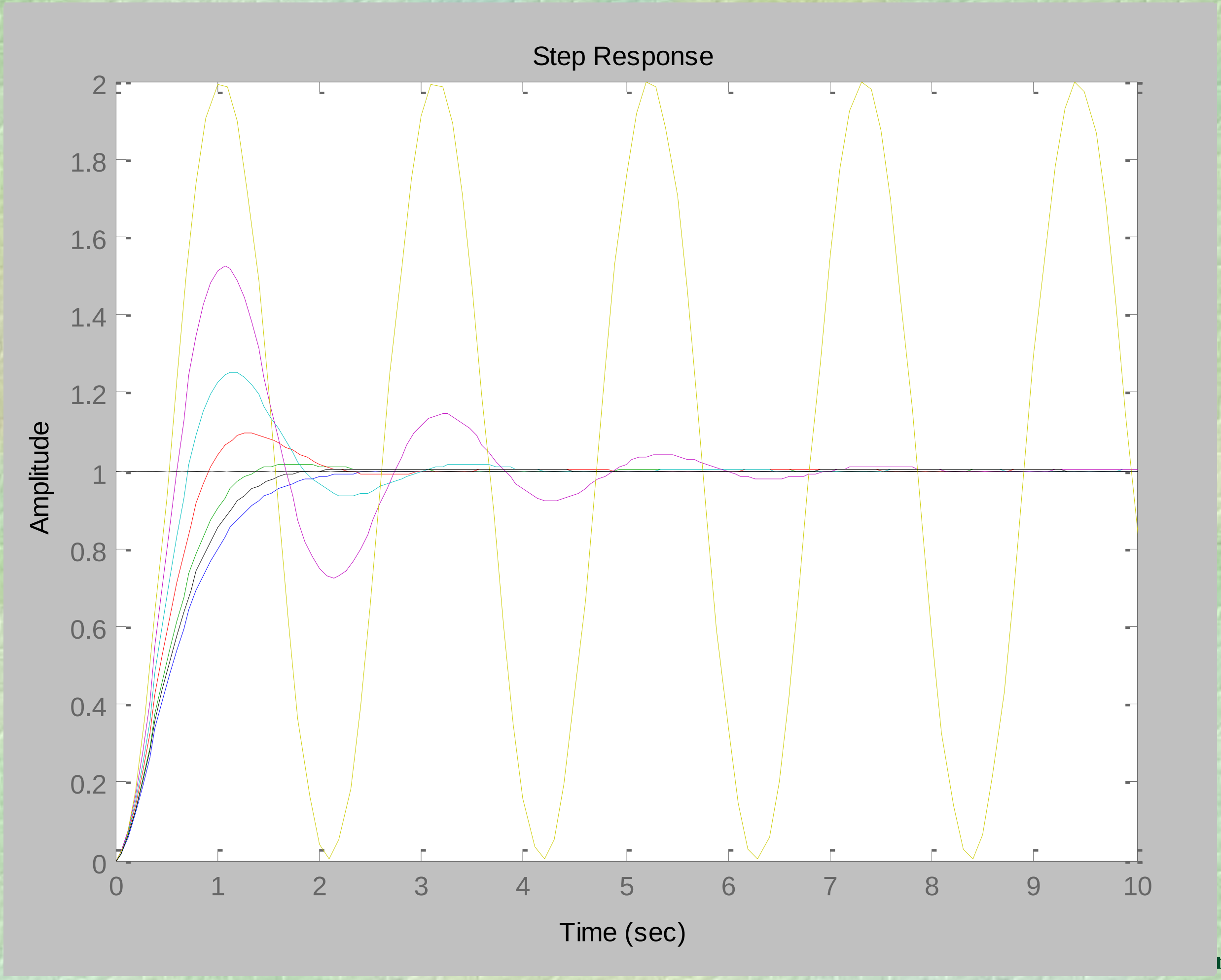
$$c(t)=u(t)\left(1-\frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}}\left(\sin\theta\cos(\omega_n\sqrt{1-\zeta^2}\,t)+\cos\theta\sin(\omega_n\sqrt{1-\zeta^2}\,t)\right)\right)$$

$$c(t)=u(t)\left(1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\theta)\right) \qquad \theta=\cos^{-1}\zeta$$

Step response

پاسخ پله

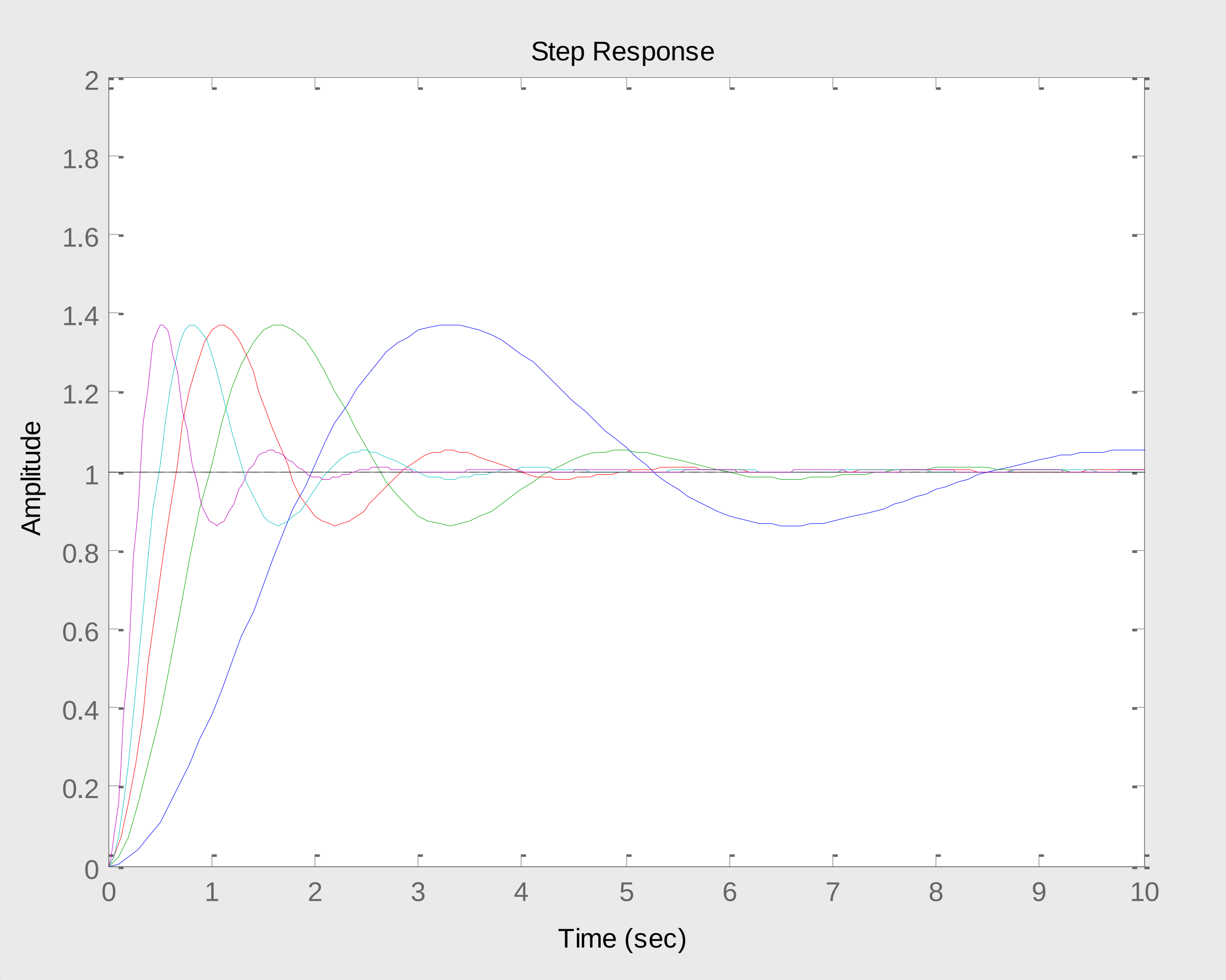
$$\omega_n = 3 \quad \zeta = 1, 0.8, 0.6, 0.4, 0.2, 0$$



Step response

پاسخ پله

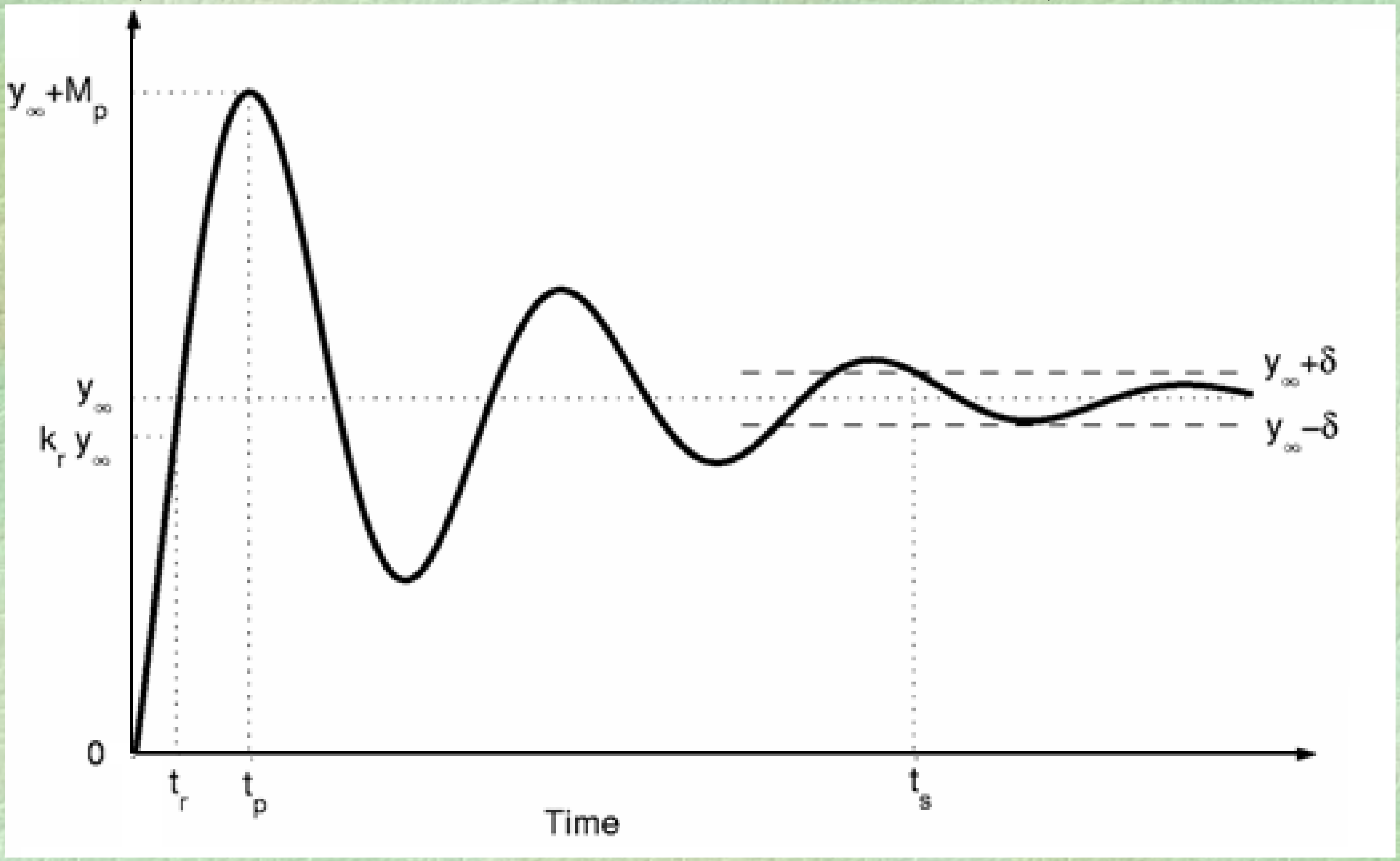
$$\zeta = 0.3 \quad \omega_n = 1, 2, 3, 4, 6.28$$



Specifications of a prototype second order system.

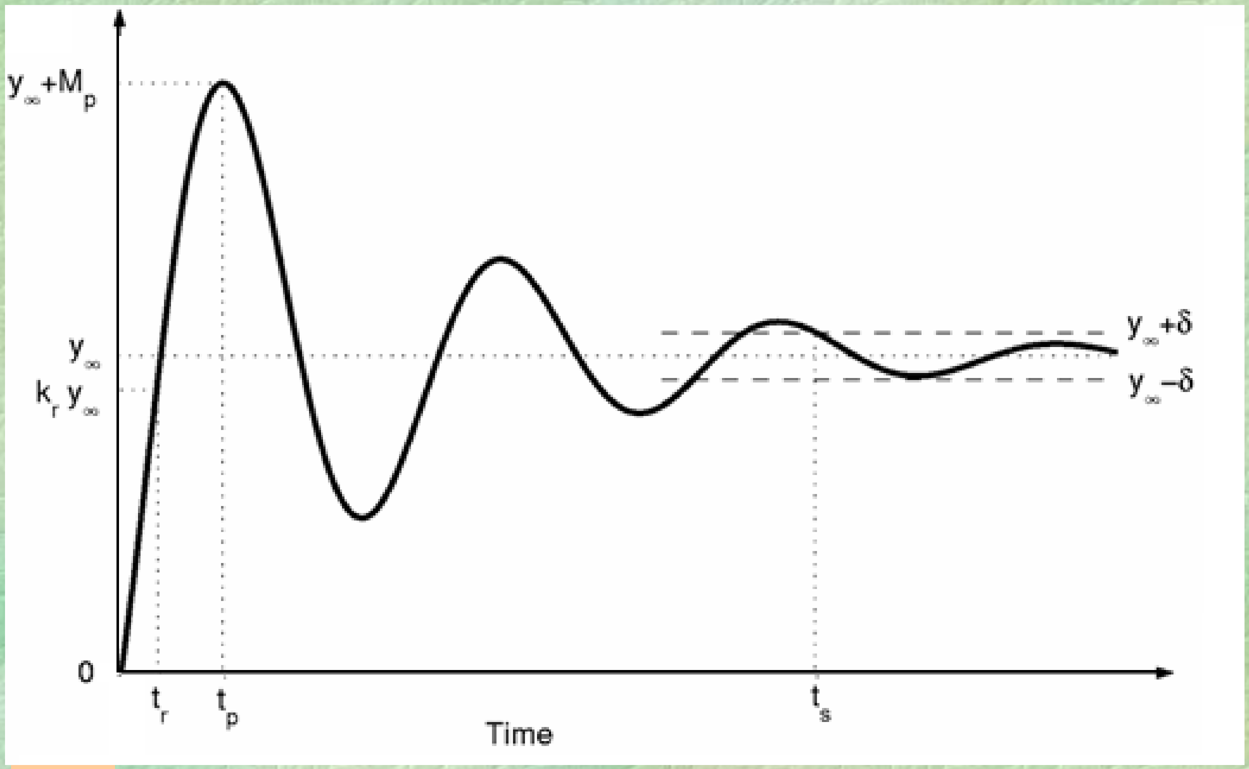
مشخصه های یک سیستم نمونه درجه ۲

$$c(t)=u(t)\left(1-\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t}\sin(\omega_n\sqrt{1-\zeta^2}\,t+\theta)\right)\qquad \theta=\cos^{-1}\zeta$$



Rise time

زمان صعود

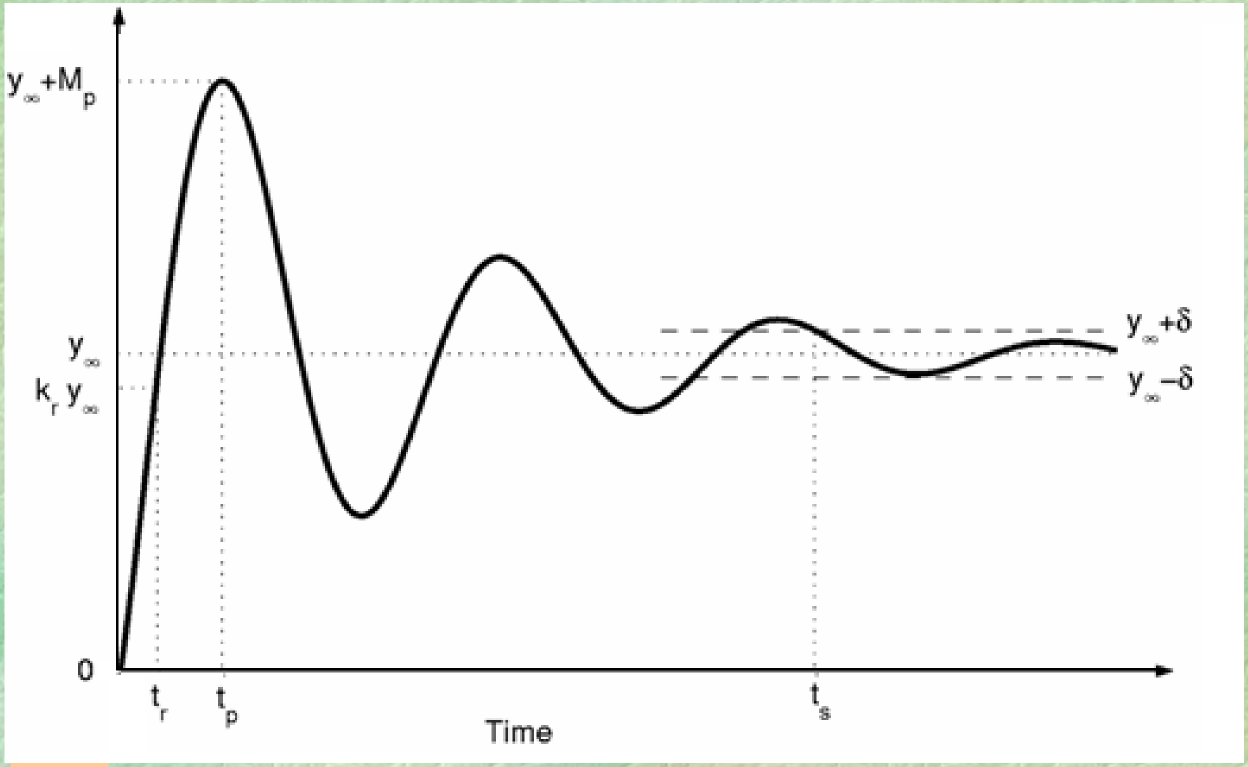


t_r: زمان سپری شده تا رسیدن پاسخ پله برای اولین بار به مقدار $k_r y_\infty$ که ثابت k_r را معمولا 0.9 یا 1 در نظر می گیرند.

t_r : The time elapsed up to the instant at which the step response reaches, for the first time, the value $k_r y_\infty$. The constant k_r varies from author to author, being usually either 0.9 or 1.

Settling time

زمان نشست

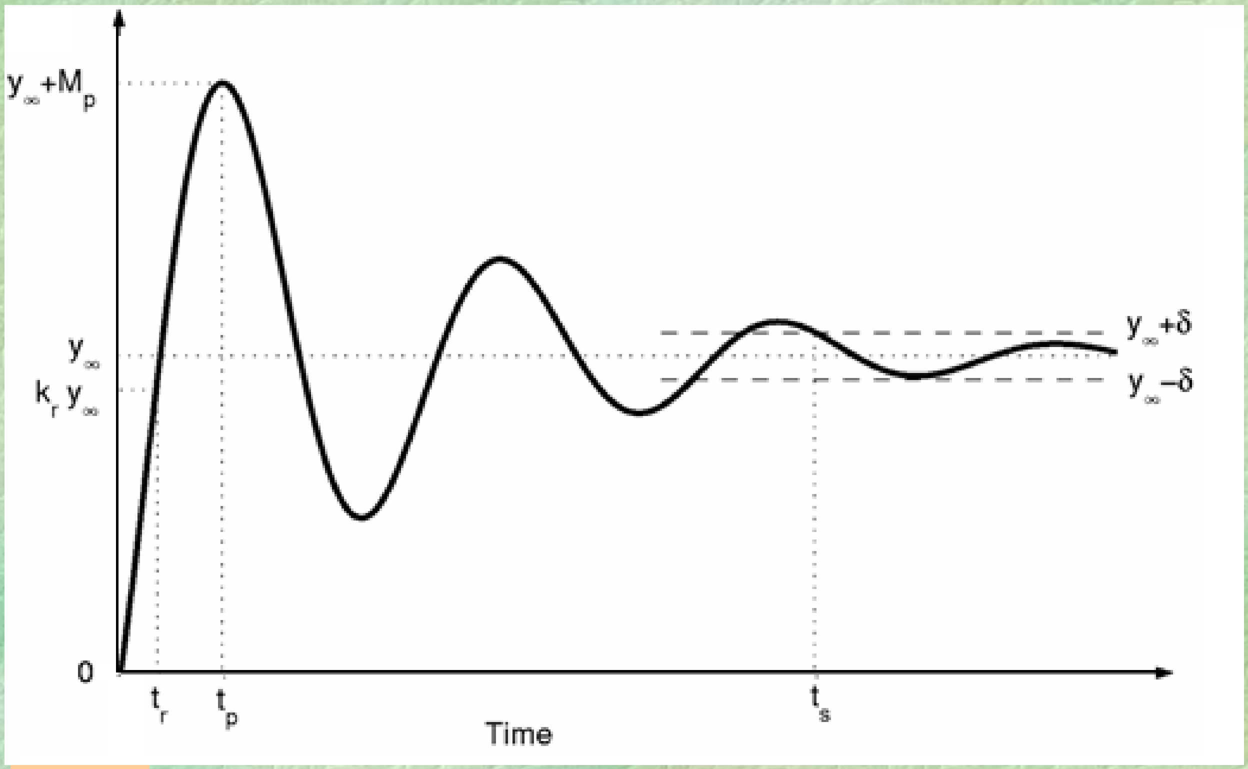


t_s: زمان سپری شده تا رسیدن پاسخ پله به ناحیه ±δ حول مقدار نهایی پاسخ. که ثابت δ را معمولا 2% یا 5% در نظر می گیرند.

t_s : The time elapsed until the step response enters (without leaving it afterwards) a specified deviation band, ±δ, around the final value. This deviation δ, is usually defined as a percentage of y_∞, say 2% to 5%.

Overshoot

فراجهش

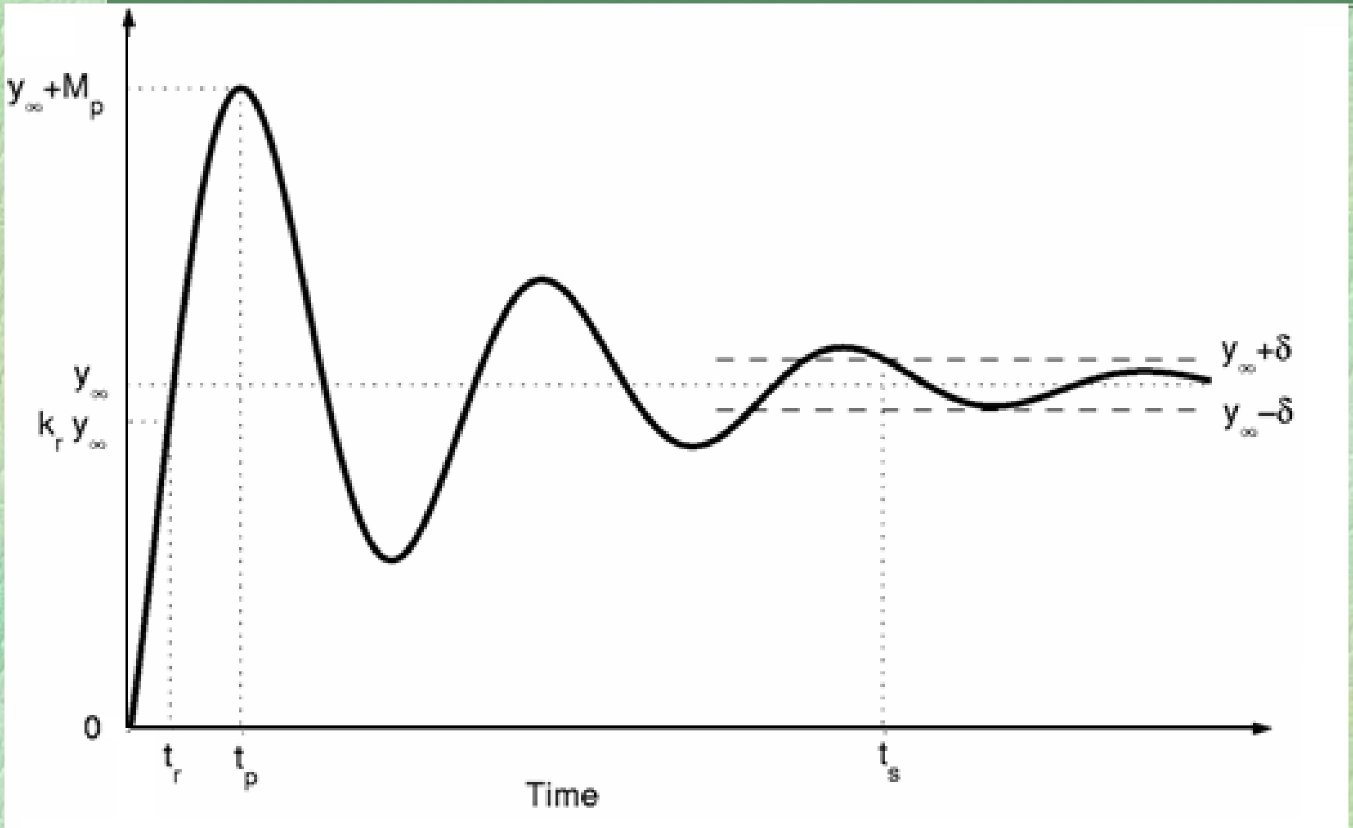


M_p : ماکزیمم مقداری که پاسخ پله از مقدار نهائی پاسخ اضافه می شود.

M_p : The maximum instantaneous amount by which the step response exceeds its final value.

Peak time

زمان پیک

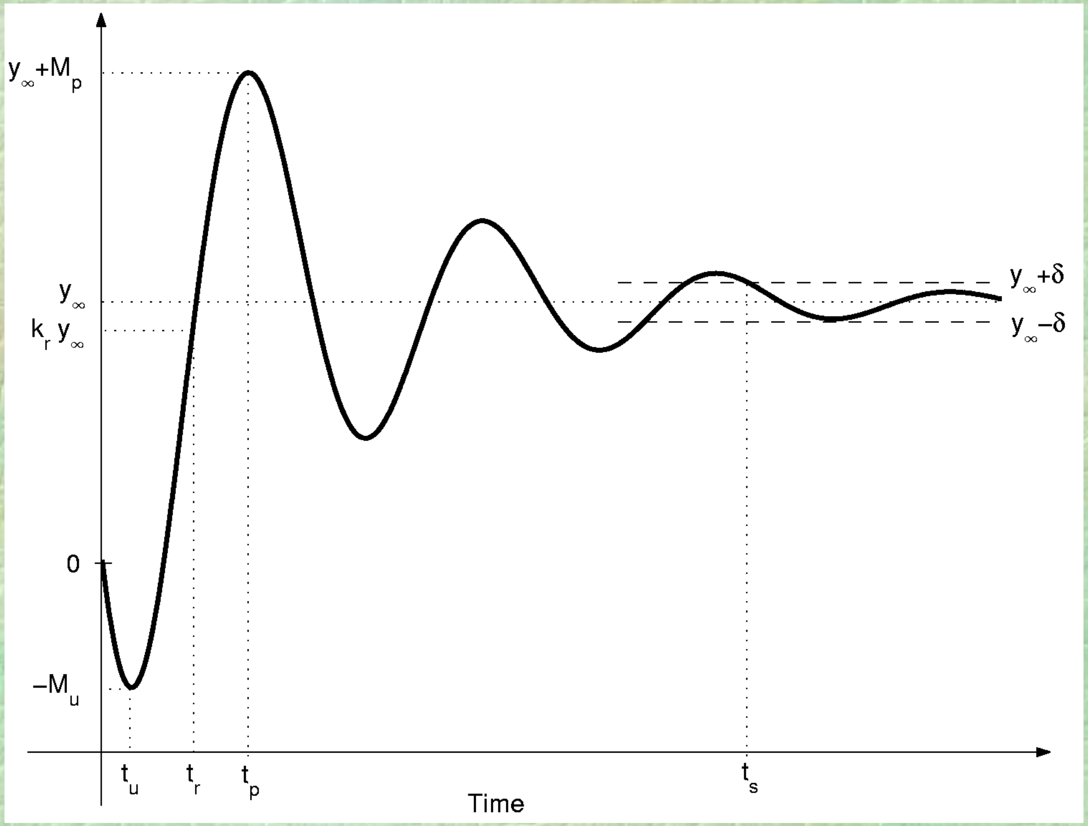


$:t_p$ زمان متناظر با اولین ماکزیمم پاسخ.

t_p : The time at which corresponding to maximum instantaneous amount by which the step response exceeds its final value.

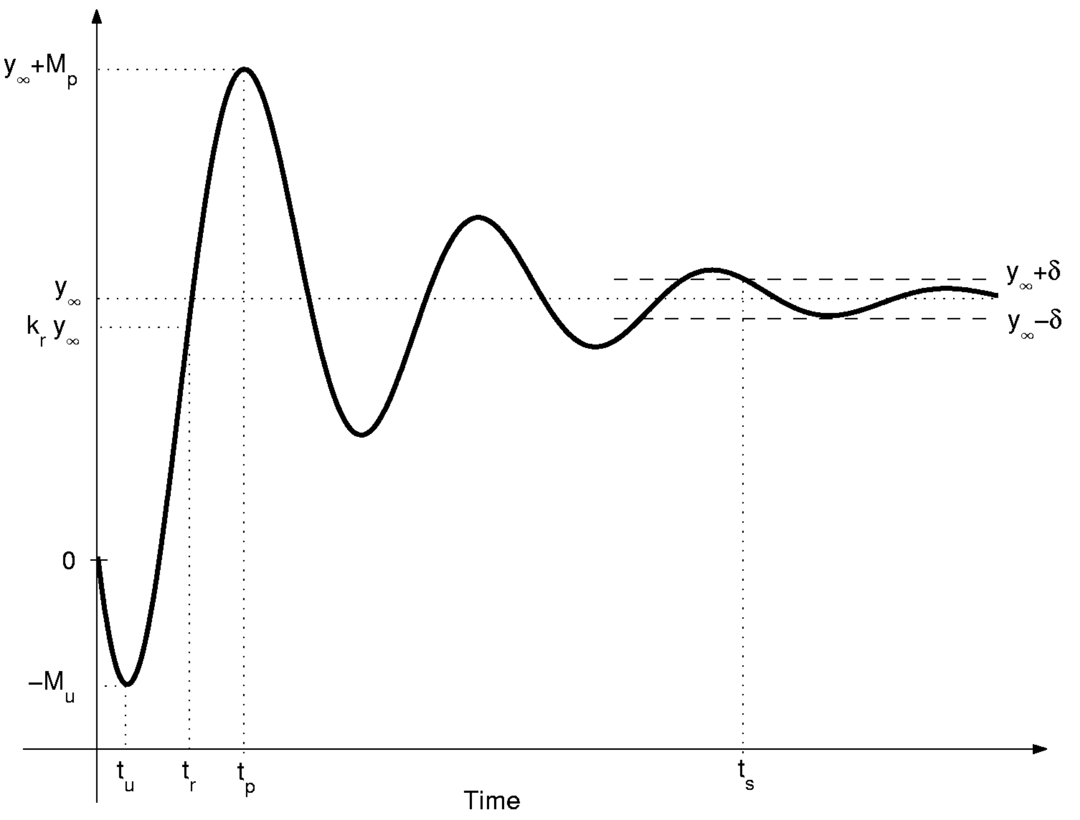
If the closed loop system includes an RHP zero

اگر سیستم حلقه بسته دارای صفر RHP باشد



Undershoot

فرو جهش

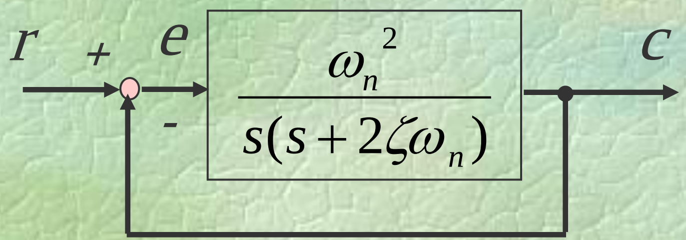


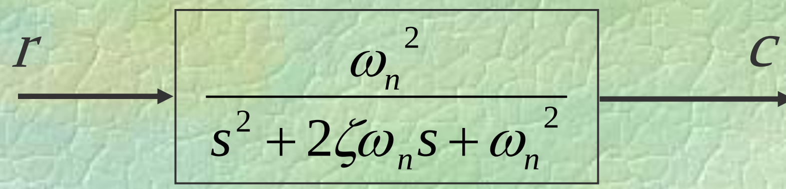
M_u: ماکزیمم مقداری (قدر مطلق) پاسخ پله از صفر کمتر می شود.

M_u : The (absolute value of the) maximum instantaneous amount by which the step response falls below zero.

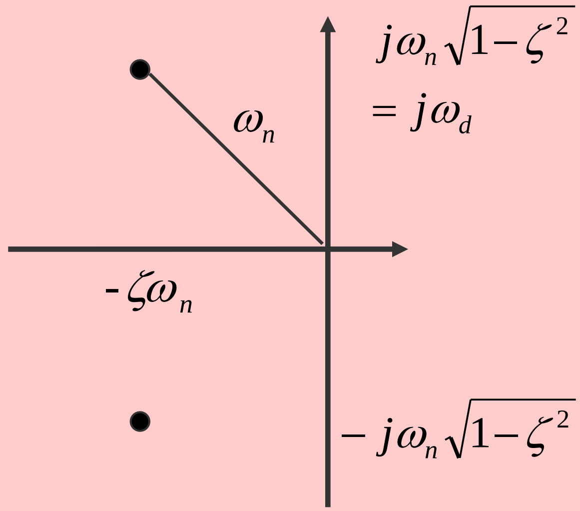
Introducing a prototype second order system.

معرفی یک سیستم نمونه درجه ۲





Poles are : $-\zeta\omega_n \pm j\omega_n\sqrt{1-\zeta^2} = -\zeta\omega_n \pm j\omega_d$ if $0 \le \zeta \le 1$



ω_n Natural frequency فرکانس طبیعی

$\zeta\omega_n$ Damping factor ضریب میرائی

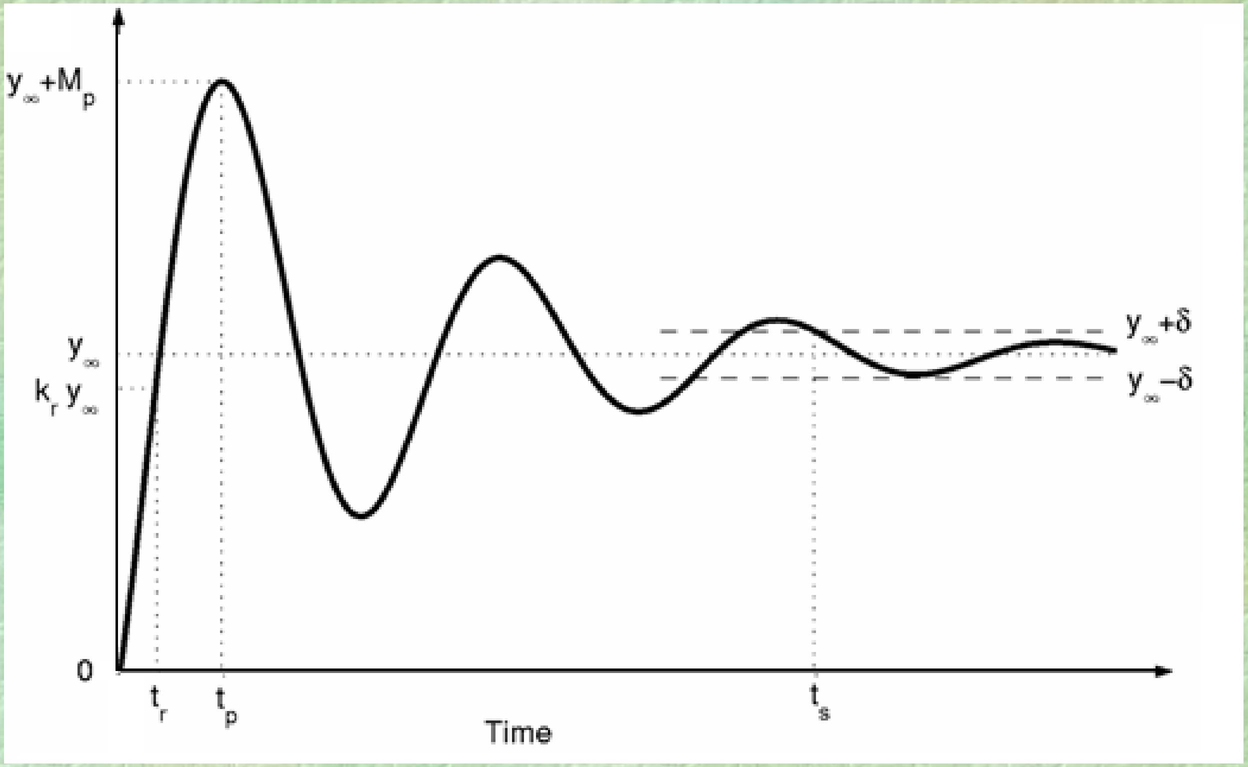
ζ Damping ratio نسبت میرائی

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ Natural damped frequency

فرکانس طبیعی میرا شده

Percent Overshoot

درصد فراجهش



$$P.O. = \% \frac{M_p}{y_\infty} 100$$

How can we find P.O. ?

چگونه درصد فراجهش را بیابیم؟

Calculation of Percent Overshoot and Peak Time

محاسبه درصد فراجهش و لحظه پیک

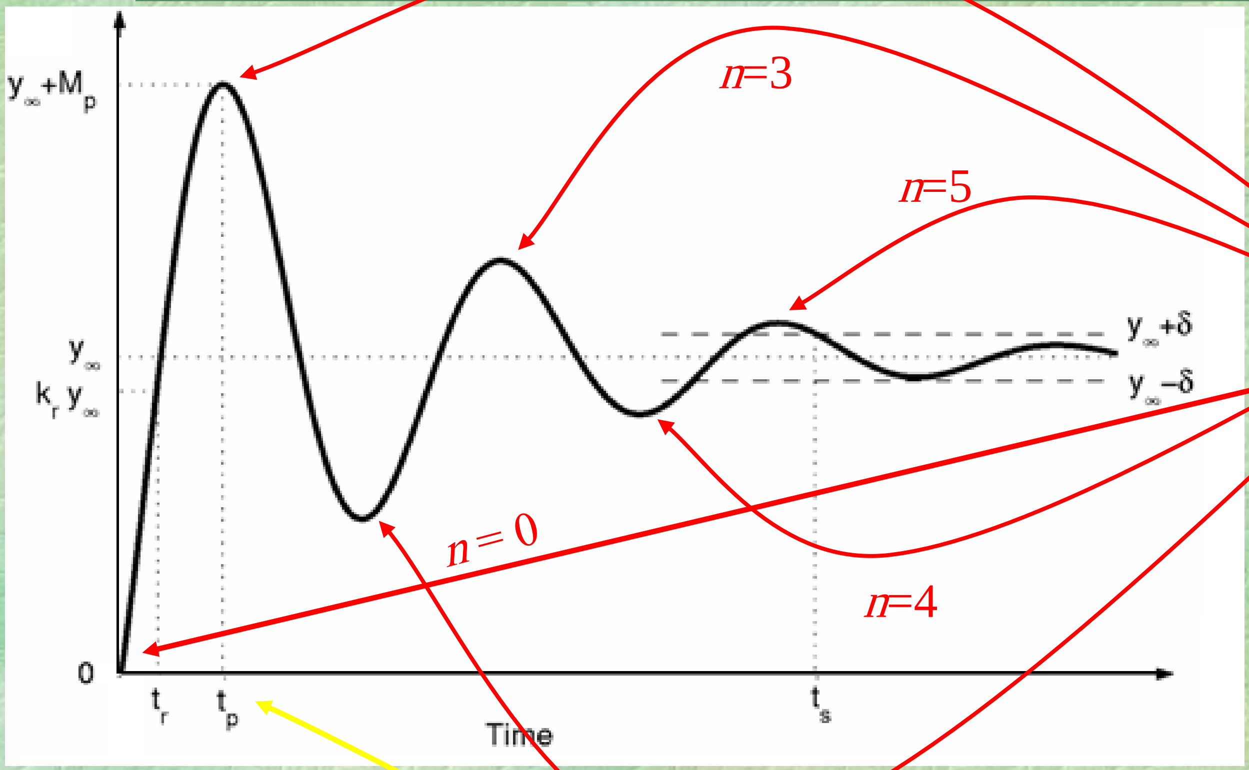
$$c(t) = u(t)\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta)\right) \qquad \theta = \cos^{-1}\zeta$$

$$\frac{\partial c(t)}{\partial t} = \frac{\zeta\omega_n}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n \sqrt{1-\zeta^2}\, t + \theta) - \omega_n e^{-\zeta\omega_n t} \cos(\omega_n \sqrt{1-\zeta^2}\, t + \theta) = 0$$

$$\tan(\omega_n \sqrt{1-\zeta^2}\, t + \theta) = \frac{\sqrt{1-\zeta^2}}{\zeta} = \tan\theta \qquad \omega_n \sqrt{1-\zeta^2}\, t = n\pi$$

Peak time

لحظه پیک



$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi$$

$$t = \frac{n\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Let n=1

$$t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

Calculation of Percent Overshoot

محاسبه درصد فراجهش

$$c(t) = u(t)\left(1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_n\sqrt{1-\zeta^2}\,t + \theta)\right) \qquad \theta = \cos^{-1}\zeta$$

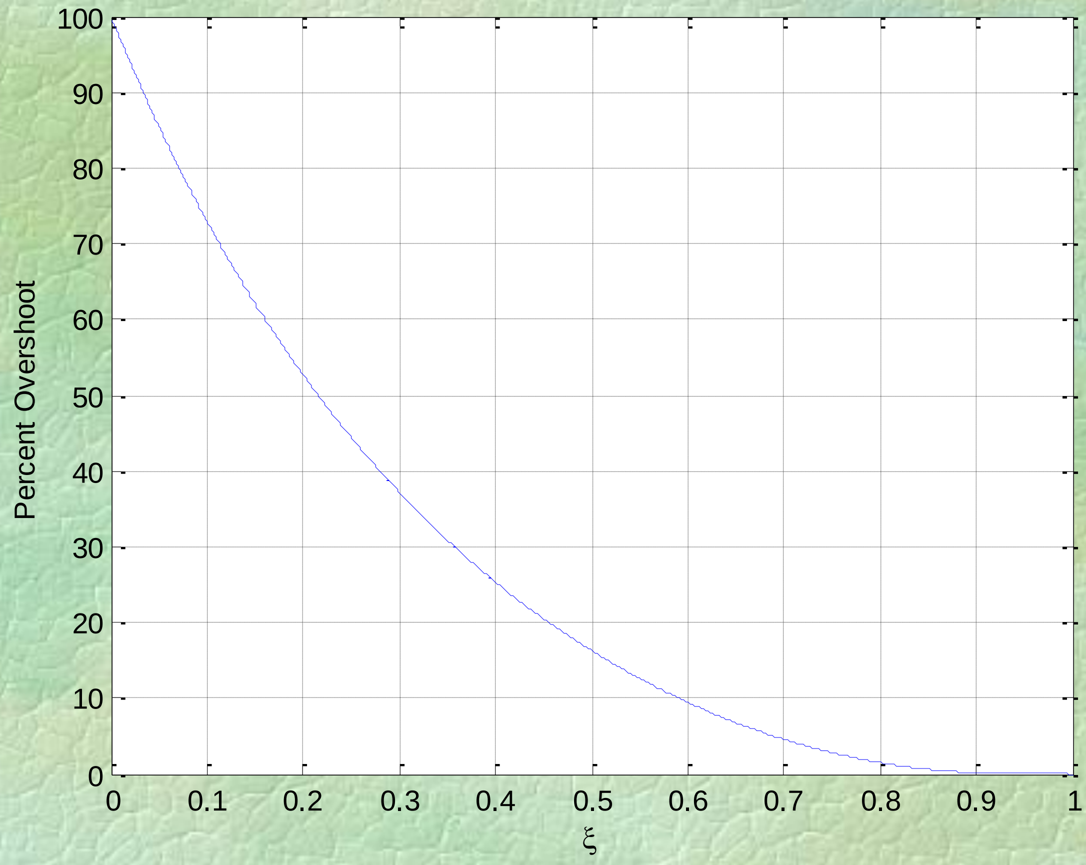
$$\omega_n\sqrt{1-\zeta^2}\,t = n\pi \qquad t_p = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

$$y_\infty + M_p = c(t_p) \quad = 1 + e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}} \qquad P.O. = \%\frac{M_p}{y_\infty}100 \qquad P.O. = \%100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$

Percent Overshoot

درصد فراجهش

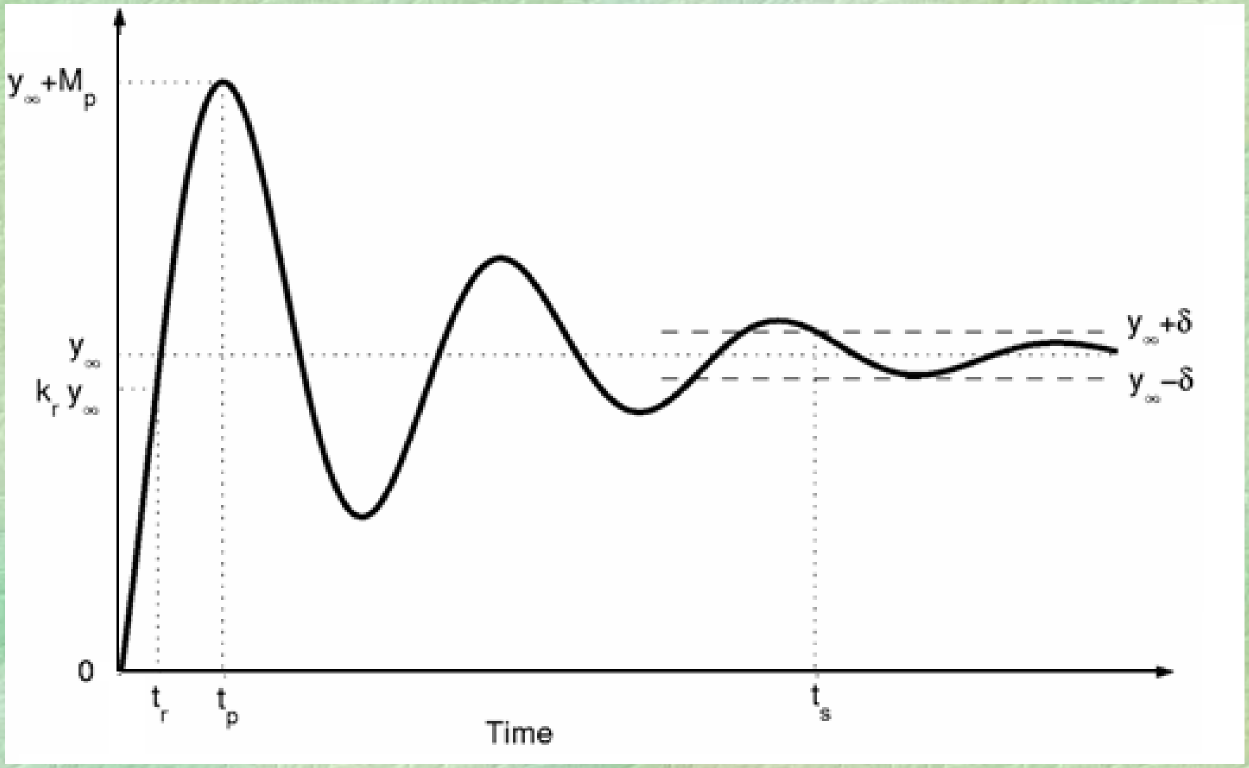
$$P.O. = \%100 e^{\frac{-\pi\zeta}{\sqrt{1-\zeta^2}}}$$



ζ	$P.O.$
0	100%
0.100	73%
0.200	53%
0.300	37%
0.400	25%
0.500	16%
0.707	4.3%
1	0%

Rise time

زمان صعود



t_r

$c(t) = k_r y_\infty$

Let $c(t) = 0.9 y_\infty$

How can we find rise time ?

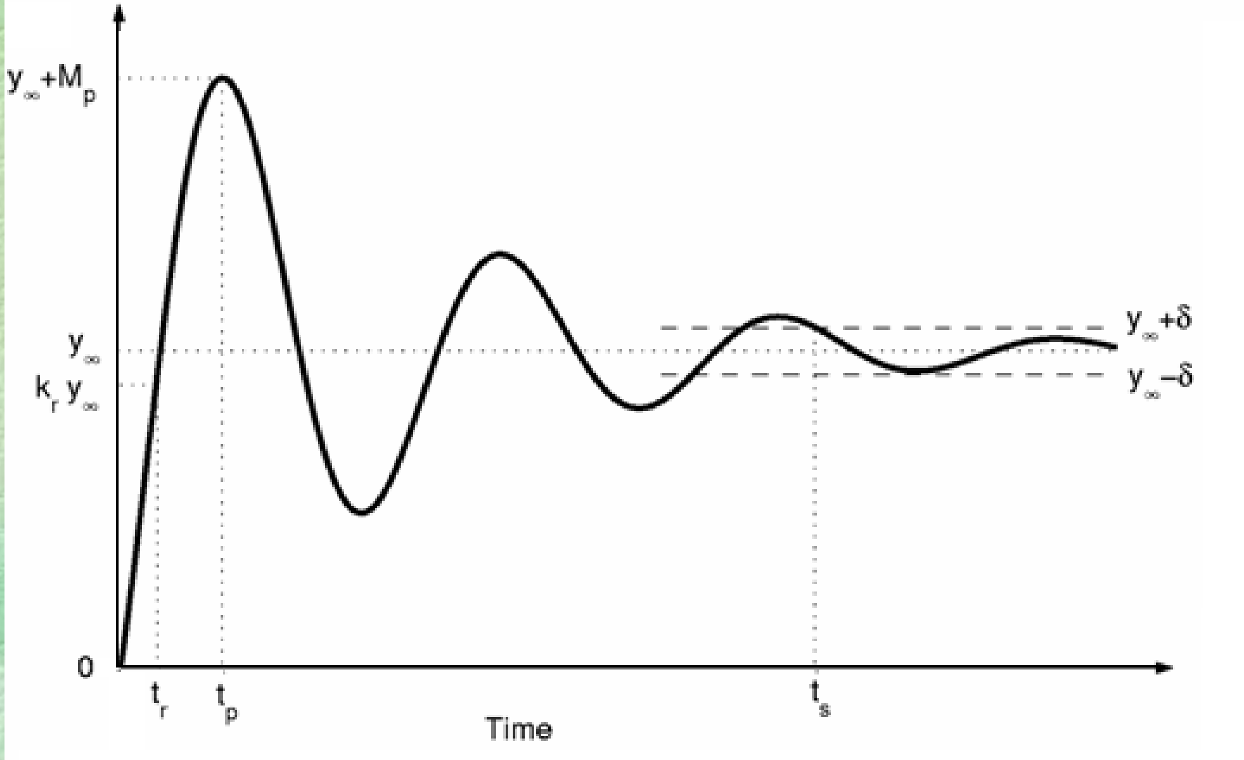
چگونه زمان صعود را بیابیم؟

$$t_r = \frac{0.8 + 2.5\zeta}{\omega_n}$$

$$t_r = \frac{1 - 0.4167\zeta + 2.917\zeta^2}{\omega_n} \qquad 0 < \zeta < 1$$

Settling time

زمان نشست



t_s

$$y_\infty - \delta \le |c(t)| \le y_\infty + \delta$$

How can we find settling time ?

چگونه زمان نشست را بیابیم؟

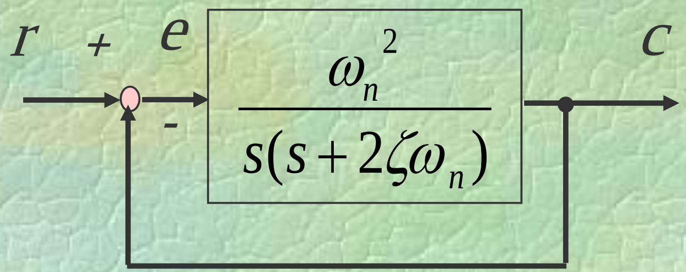
$$t_s = \frac{3.2}{\zeta\omega_n} \quad \text{for} \quad \delta = 5\%$$

$$t_s = \frac{4}{\zeta\omega_n} \quad \text{for} \quad \delta = 2\%$$

$$0 < \zeta < 0.7$$

Exercises

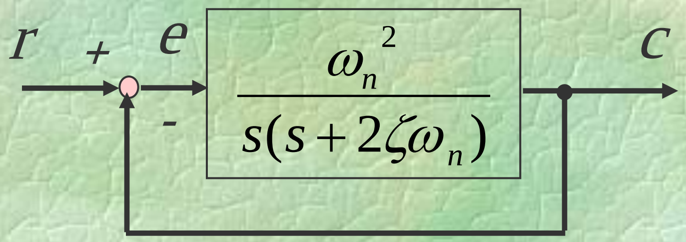
1 – Consider following system.



a) Find the step response of the system for $\omega_n = 12.56\ ,\ \zeta = 0.3$

b) Find the rise time, settling time, overshoot, and percent overshoot.

2 – Consider following system.

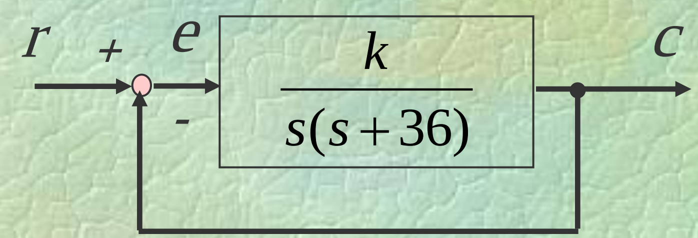


a) Find the step response of the system for $\omega_n = 12.56\ ,\ \zeta = 0.9$

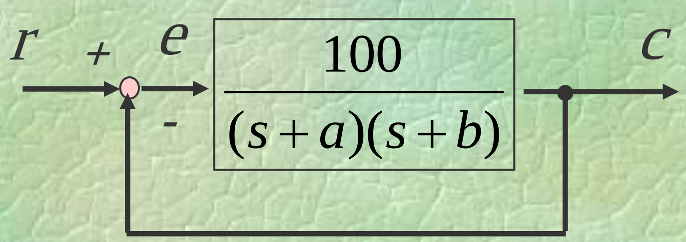
b) Find the rise time, settling time, overshoot, and percent overshoot.

Exercises (Cont.)

3- In the following system set k such that the percent overshoot of system be 4.3%



4- In the following system set a and b such that the percent overshoot of system be 4.3%
And the steady state error to step input be 0.



5- In the system of problem 1 set k such that

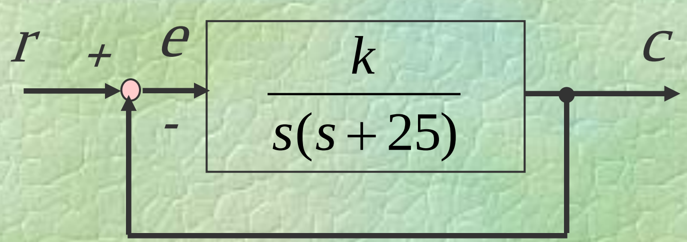
a) The error to ramp input be 0.01
b) The percent overshoot of system be 4.3%
c) The error to ramp input be 0.01 and the percent overshoot of system be 4.3%

Exercises (Cont.)

6- In the following system

a) For k=200 derive settling time, rise time and percent overshoot. Confirm your result with step response.

b) For k=1000 derive settling time and percent overshoot. Confirm your result with step response.



7- In the following system set the k such that the imaginary poles have 0.707 damping ratio.