LINEAR CONTROL SYSTEMS

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Lecture 15

Root Locus Technique

Topics to be covered include:

- Property and construction of complete root loci. (Cont.)
- Effect of adding poles and zeros on root locus.
- Effect of moving poles and zeros.
- Root contour.

Example 1: Draw the complete root loci of following system.

مثال ۱: مکان کامل ریشه ها را در سیستم زیر بیابید.

$$1+10\frac{(s+k)(s+3)}{s(s^2-1)}=0$$

Rule 1: Specify the equation **exactly** in the standard form.

قانون اول: سیستم را **دقیقا** بصورت زیر استاندارد کنید.

$$s(s^2-1)+10(s+k)(s+3)=0$$

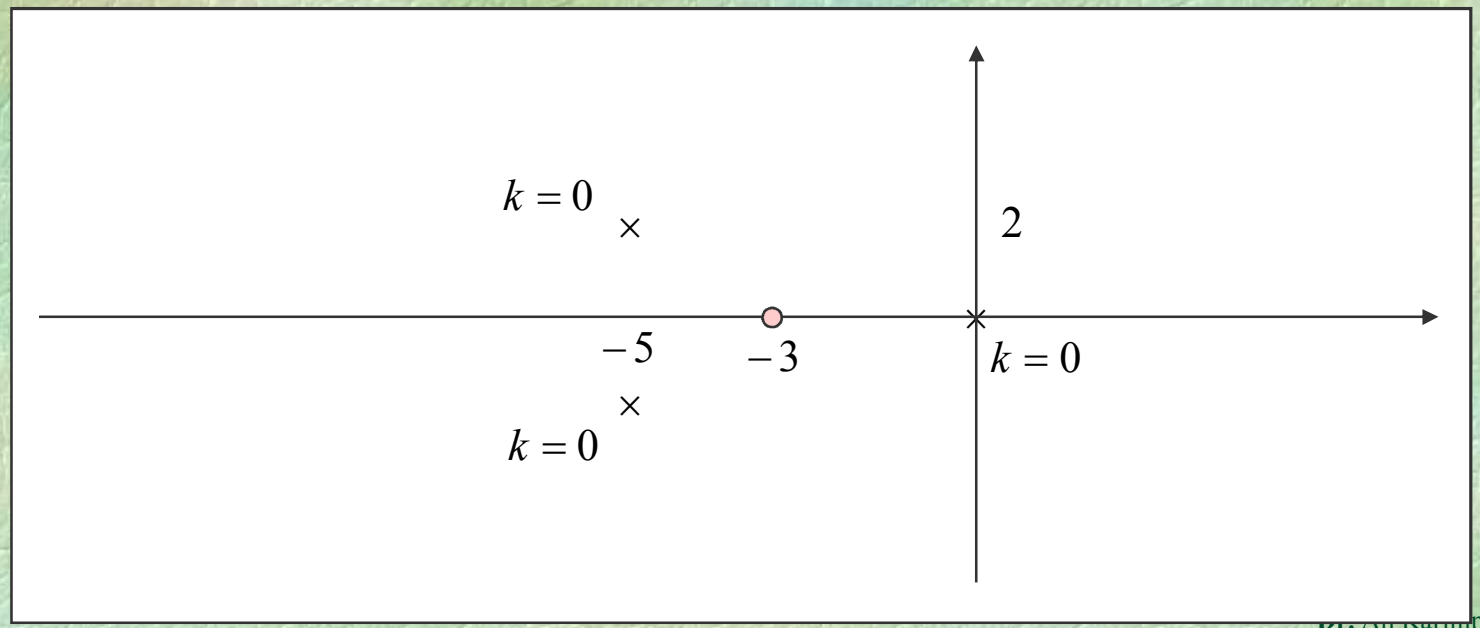
$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

Example 1:

$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

Rule 2: Specify the poles and zeros of f(s). The root loci lie on the poles of f(s) for k=0 and lie on the zeros of f(s) for k=±∞

قانون ۲: قطب و صفرهای f(s) را مشخص کنید. مکان ریشه در k=0 روی قطبهای f(s) و در k=±∞ روی صفرهای f(s) قرار دارد.

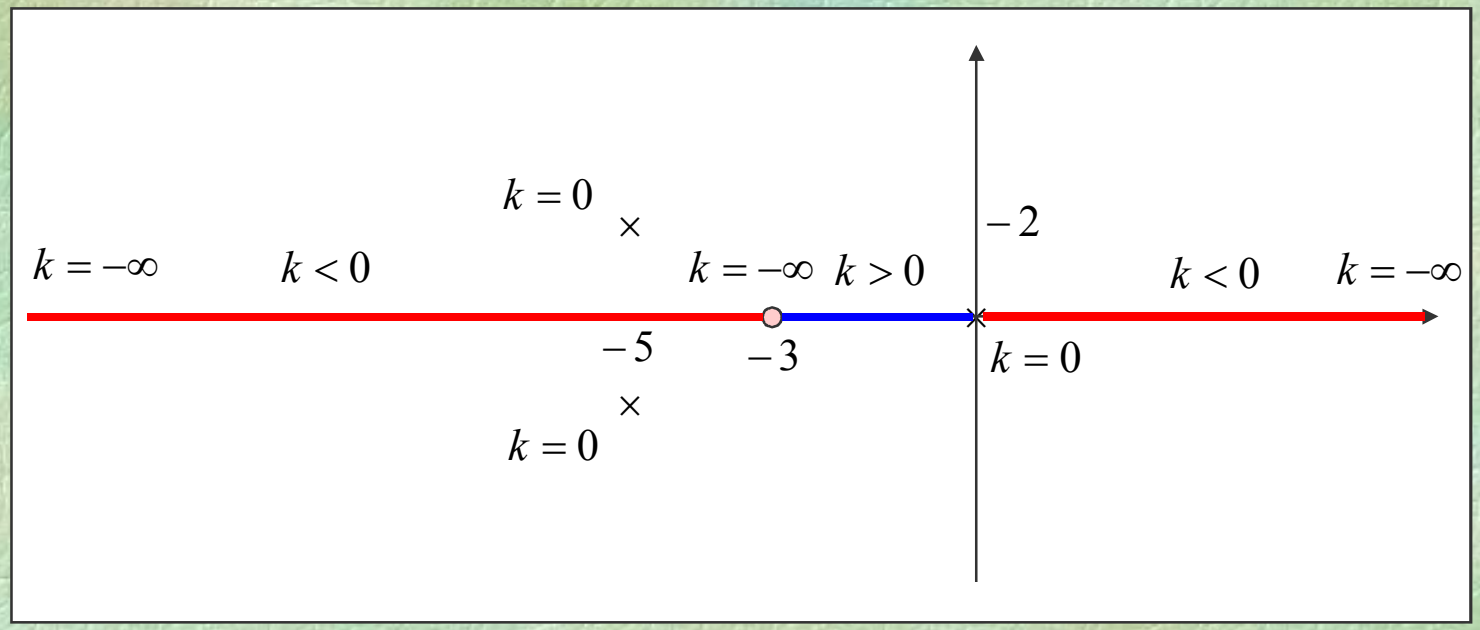


Example 1:

$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

Rule 3: Define the real axis section for positive and negative value of *k*.

قانون ۳: محور حقیقی را برای مقادیر مثبت و منفی k مشخص کنید



Example 1:

$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

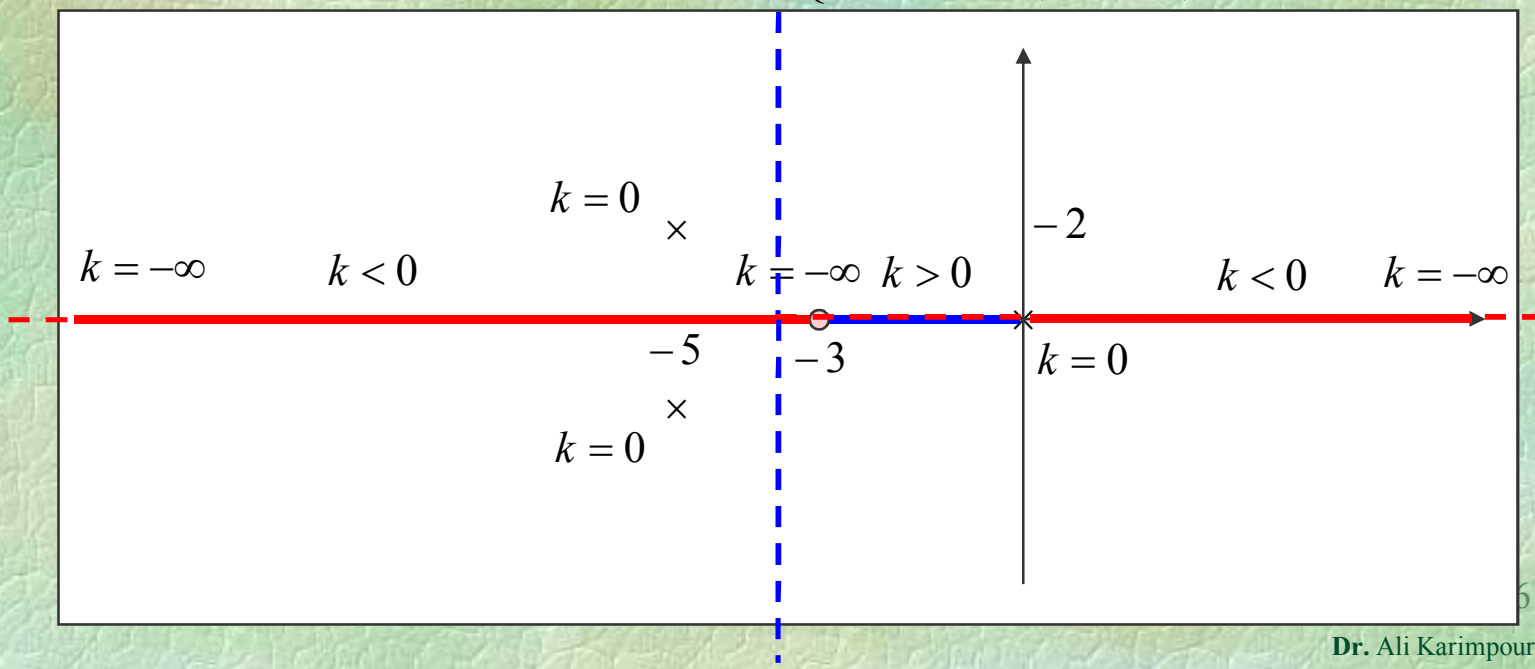
Rule 4: Find the asymptotes and centered of asymptotes .

قانون ۴: مجانبها و محل تلاقی مجانبها.

Asymptotes center

$$\delta=\frac{\sum_{i=1}^{n_p}p_i-\sum_{i=1}^{n_z}z_i}{n_p-n_z}=\frac{-10-(-3)}{2}=-3.5$$

$$\begin{cases}\text{k}>0 & \theta=\dfrac{(2m+1)\pi}{|n_p-n_z|}=\dfrac{\pi}{2},\dfrac{3\pi}{2}\\ \text{k}<0 & \theta=\dfrac{2m\pi}{|n_p-n_z|}=0,\dfrac{2\pi}{2}\end{cases}$$



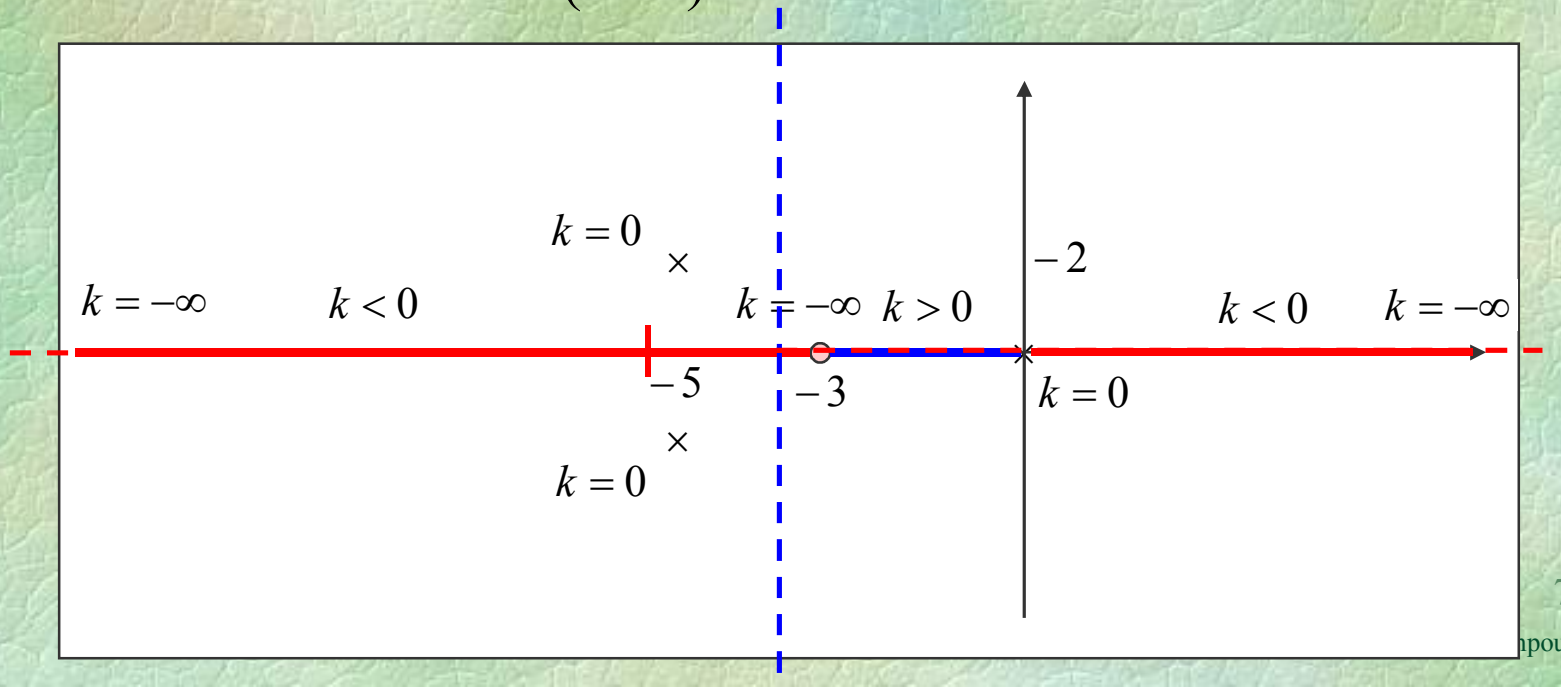
Example 1:

$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

Rule 5: Find the break point. قانون ۵: نقطه شکست را بیابید.

$$k=-\frac{1}{f(s)}=-\frac{1}{10}\frac{(s^3+10s^2+29s)}{(s+3)}$$

$$\frac{\partial k}{\partial s}=-\frac{1}{10}\frac{(3s^2+20s+29)(s+3)-(s^3+10s^2+29s)}{(s+3)^2}=0 \qquad s=-5.47$$



$$1+k\frac{10(s+3)}{s(s^2+10s+29)}=0$$

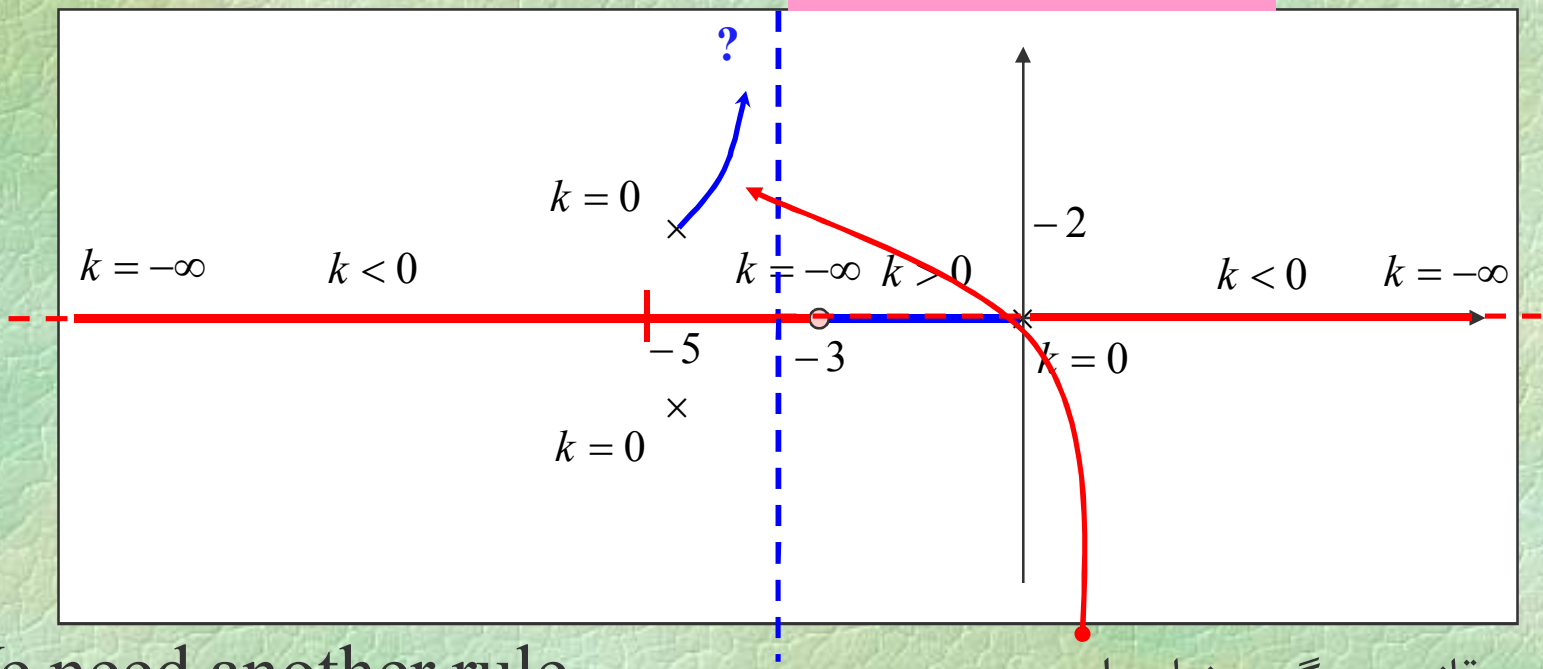
قانون ۶: نقطه تلاقی با محور موهومی

Rule 6: Find the cross of root locus with imaginary axis

$$s^3+10s^2+(29+10k)s+30k=0$$

s^3	1	$29+10k$
s^2	10	$30k$
s^1	$29+7k$	0
s^0	$30k$	

$$\Rightarrow \quad k>0$$



We need another rule.

به قانون دیگری نیاز داریم.

The Root Locus procedure

نحوه رسم مکان ریشه ها

$$1 + kf(s) = 0$$

Rule 7: Find the arrival angles and departure angles.

قانون ۷: زوایای ورود و خروج را تعیین کنید.

Departure angles.

$$\sum \angle \text{Zeros} - \sum \angle \text{poles} = \pi$$

Arrival angles.

$$\sum \angle \text{Zeros} - \sum \angle \text{poles} = 0$$

Rule 7: Find the arrival angles and departure angles.

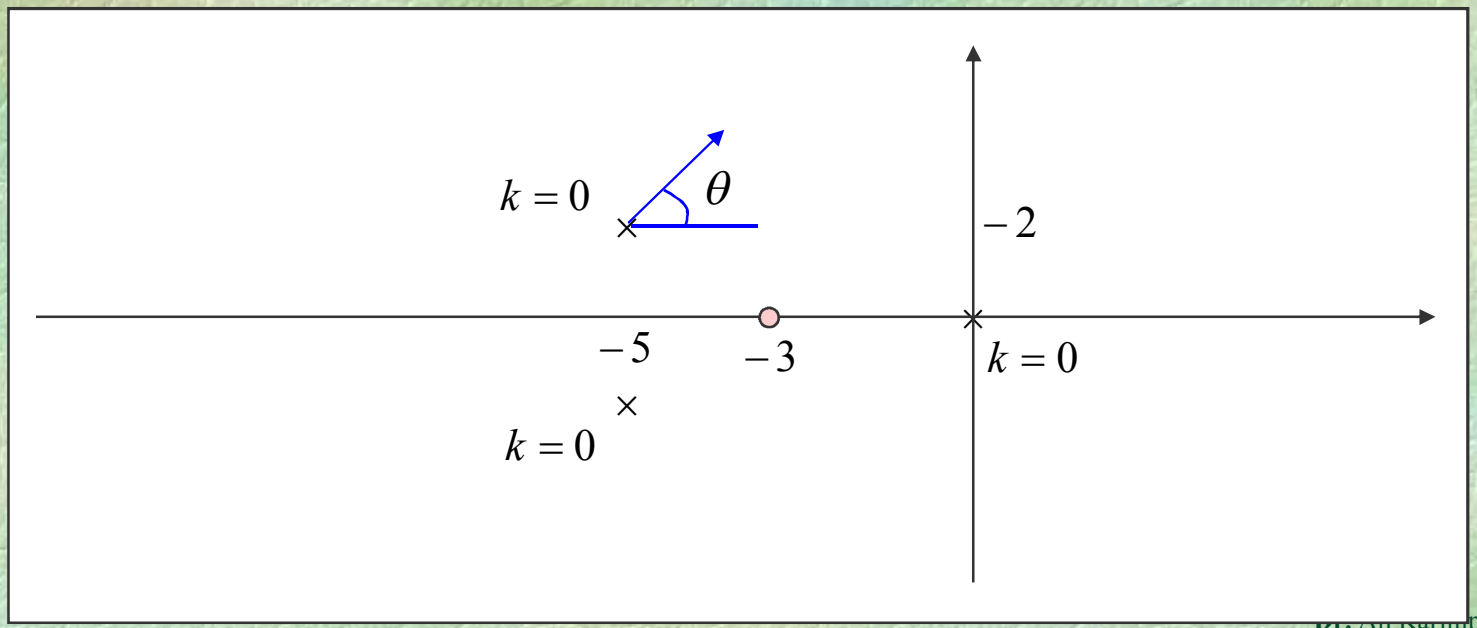
قانون ۷: زوایای ورود و خروج را تعیین کنید.

$$135 - \theta - 90^\circ - (180^\circ - \tan^{-1}\frac{2}{5}) = -\theta - 113.2^\circ = 180^\circ$$

$$\theta = -293.2^\circ$$
$$= 66.8^\circ$$

$$135 - \theta - 90^\circ - (180^\circ - \tan^{-1}\frac{2}{5}) = -\theta - 113.2^\circ = 0^\circ$$

$$\theta = -113.2^\circ$$



Rule 7: Find the arrival angles and departure angles.

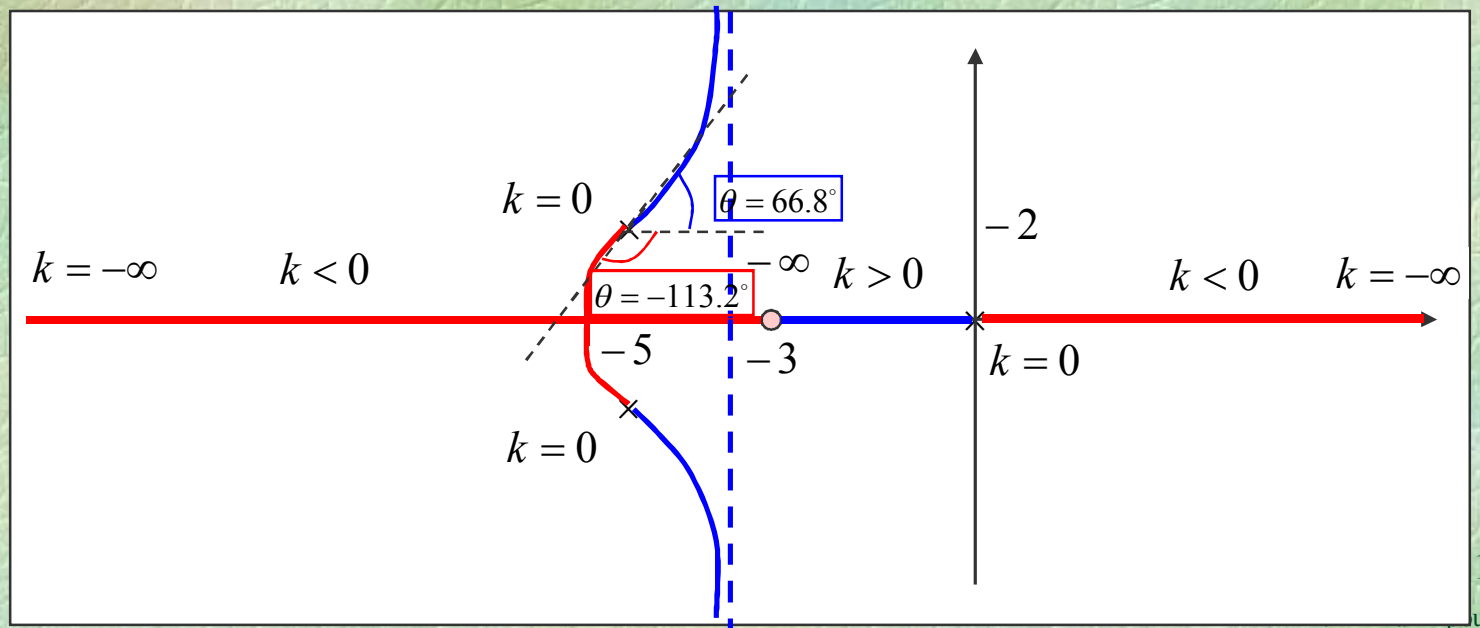
قانون ۷: زوایای ورود و خروج را تعیین کنید.

$$135 - \theta - 90^\circ - (180^\circ - \tan^{-1}\frac{2}{5}) = -\theta - 113.2^\circ = \pm 180^\circ$$

$$\theta = 66.8^\circ$$

$$135 - \theta - 90^\circ - (180^\circ - \tan^{-1}\frac{2}{5}) = -\theta - 113.2^\circ = 0^\circ$$

$$\theta = -113.2^\circ$$



Calculation of k on the Root Loci

محاسبه k روی مکان ریشه

$$1 + kf(s) = 0$$

Condition of magnitude

$$|k| = \frac{1}{|f(s)|}$$

Let $$f(s) = C \frac{\prod_{i=1}^{m} |s + z_i|}{\prod_{j=1}^{n} |s + p_j|}$$

$$|k|_{s_1} = \frac{\prod_{j=1}^{n} |s_1 + p_j|}{C \prod_{i=1}^{m} |s_1 + z_i|}$$

Summary

Rule 1: Specify the equation **exactly** in the form. $1+kf(s)=0$

Rule 2: Specify the poles and zeros of f(s). The root loci lie on the poles of f(s) for k=0 and lie on the zeros of f(s) for k=±∞

Rule 3: Define the real axis section for positive and negative value of k.

Rule 4: Find the asymptotes and centered of asymptotes .

Asymptotes center

$$\delta=\frac{\sum_{i=1}^{n_p}p_i-\sum_{i=1}^{n_z}z_i}{n_p-n_z}$$

$$\begin{cases}\text{k}>0 \quad \theta=\dfrac{(2m+1)\pi}{|n_p-n_z|}\\ \text{k}<0 \quad \theta=\dfrac{2m\pi}{|n_p-n_z|}\end{cases}$$

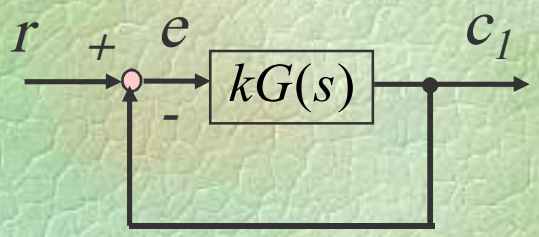
Rule 5: Find the break point.

Rule 6: Find the cross of root locus with imaginary axis by Routh Hurwitz criteria.

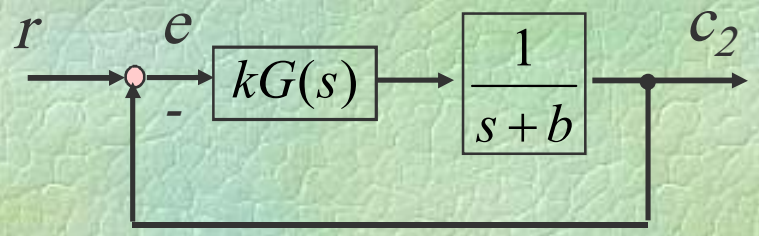
Rule 7: Find the arrival angles and departure angles.

Adding poles and zeros inside the loop

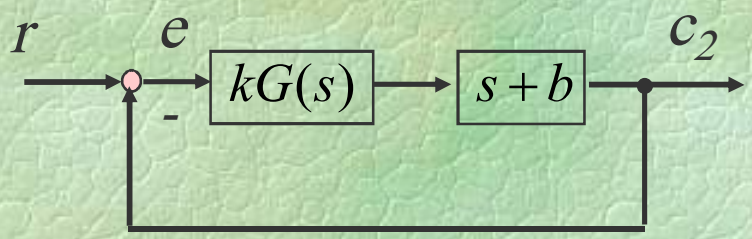
اضافه کردن صفر و قطب داخل تابع انتقال حلقه



Adding pole to the loop



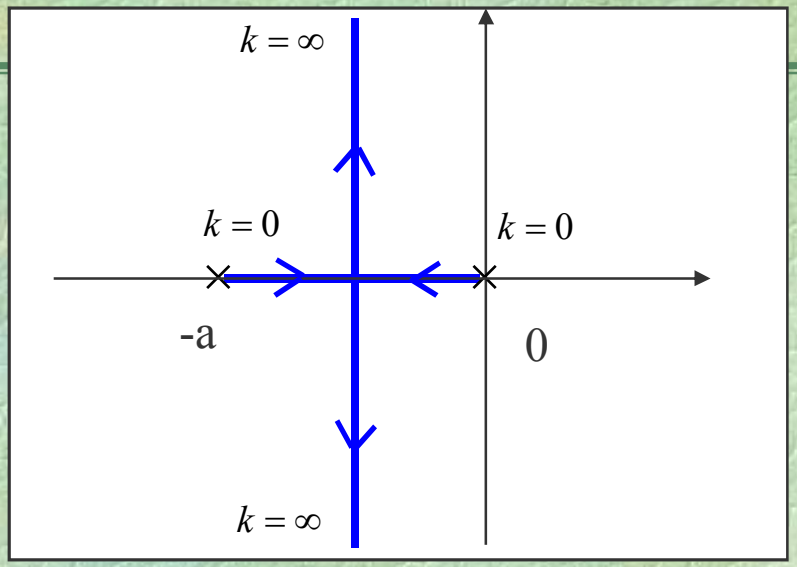
Adding zero to the loop



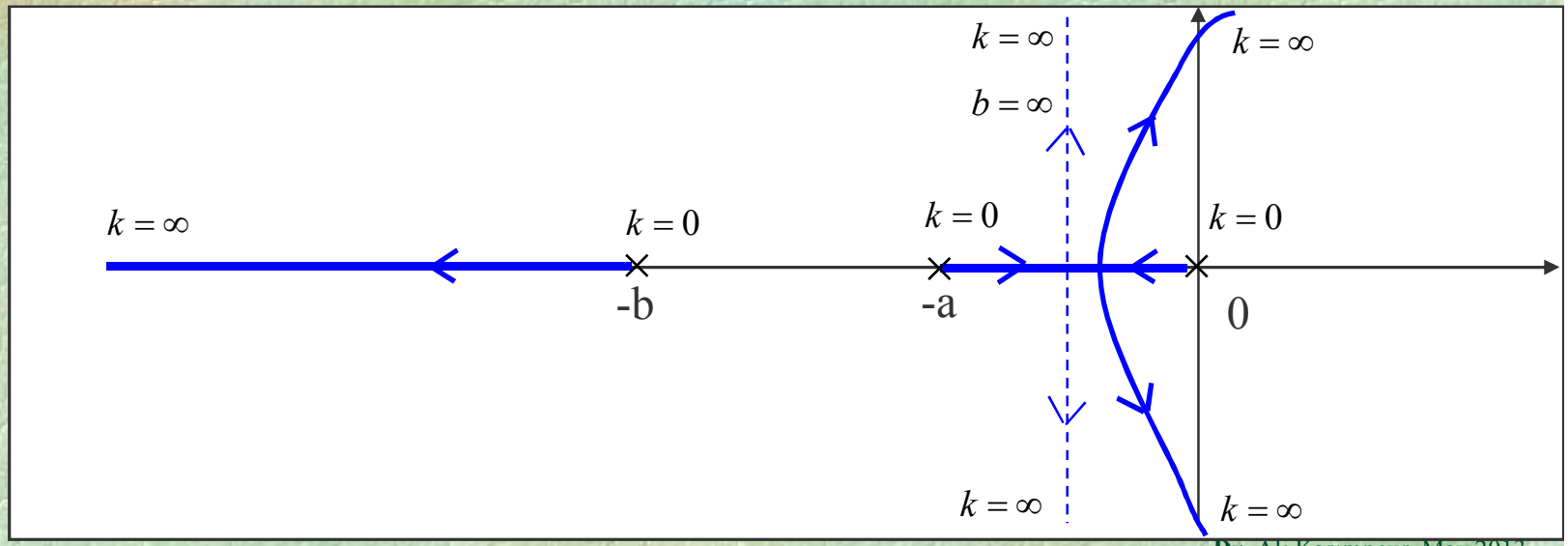
Adding poles inside the loop

اضافه کردن قطب داخل حلقه

$$1+k\frac{1}{s(s+a)}=0$$



$$1+k\frac{1}{s(s+a)(s+b)}=0$$

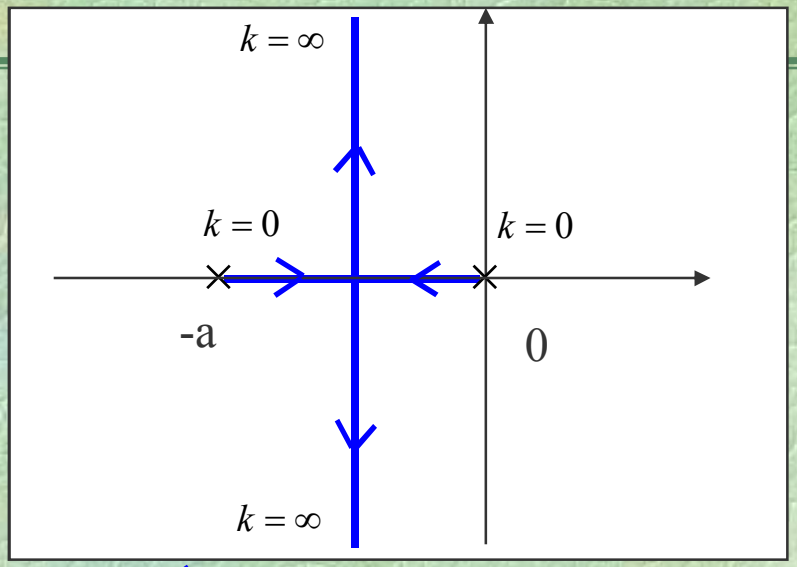


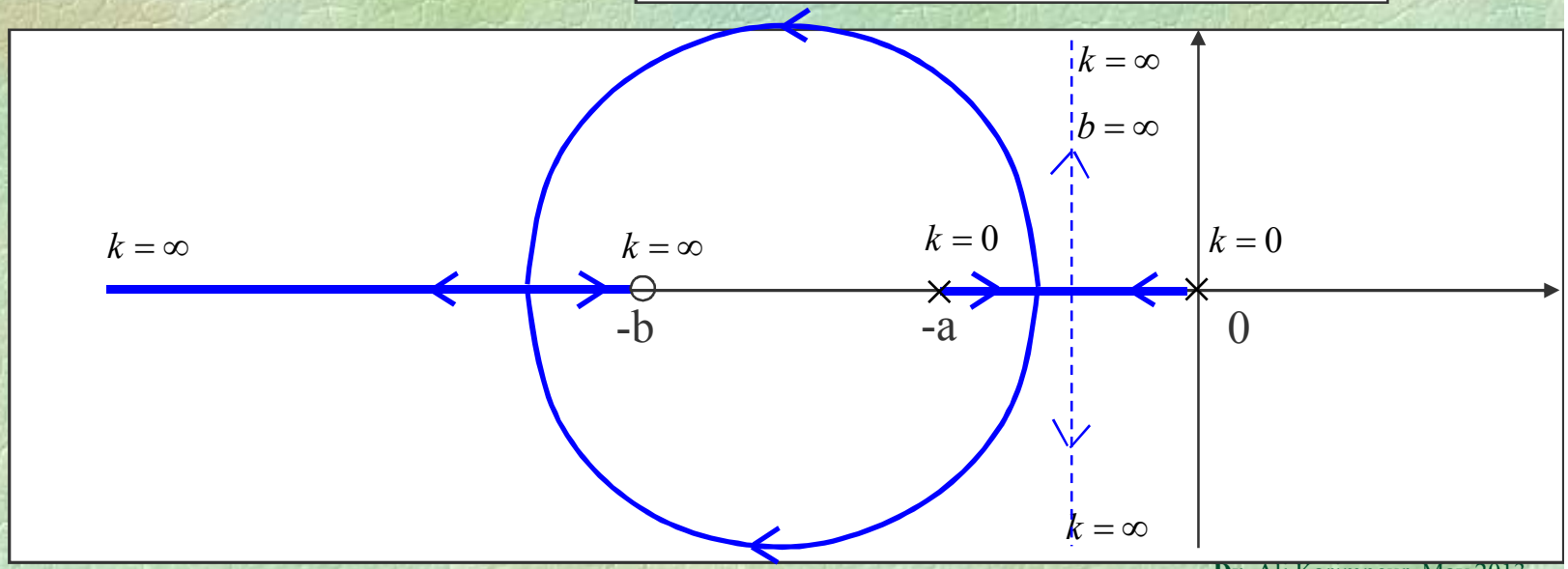
Adding zeros inside the loop

اضافه کردن صفر داخل حلقه

$$1+k\frac{1}{s(s+a)}=0$$

$$1+k\frac{(s+b)}{s(s+a)}=0$$





Effect of moving poles

اثرات حرکت قطبها

Consider following equation:

$$s^2(s+a)+k(s+b)=0 \quad \Rightarrow \quad 1+\frac{k(s+b)}{s^2(s+a)}=0$$

Let b=1 but a=10, 9, 8, 3, 1

$$1+\frac{k(s+1)}{s^2(s+a)}=0$$

Effect of moving poles.

اثرات حرکت قطبها

Let **a=10**

$$1+\frac{k(s+1)}{s^2(s+10)}=0$$

$$\delta=\frac{-10-(-1)}{2}=-4.5$$

Asymptotes are $\frac{\pi}{2},\frac{3\pi}{2}$

$$k=\frac{-s^2(s+a)}{(s+1)}$$

$$\frac{\partial k}{\partial s}=\frac{-(3s^2+2as)(s+1)+s^2(s+a)}{(s+1)^2}$$

$$(3s+2a)(s+1)=s(s+a)$$

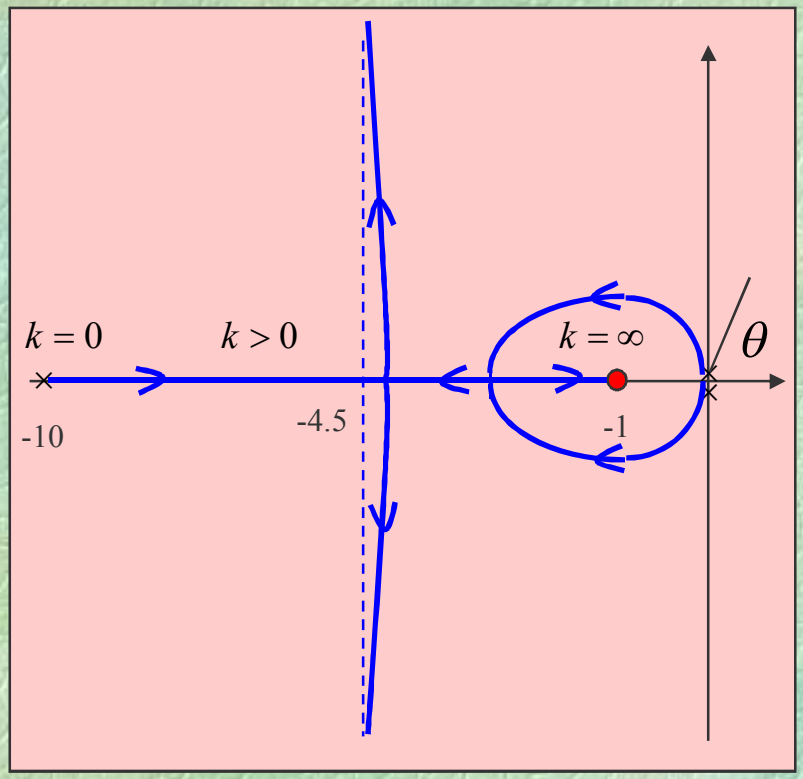
$$s^2+\frac{(a+3)}{2}s+a=0$$

$$s=-\frac{(a+3)}{4}\pm\frac{1}{4}\sqrt{a^2-10a+9}$$

Let **a=10**

$$s_1=-4\quad s_2=-2.5$$

$\theta=?$ $\theta=90$



Effect of moving poles

اثرات حرکت قطبها

Let **a=9**

$$1+\frac{k(s+1)}{s^2(s+9)}=0$$

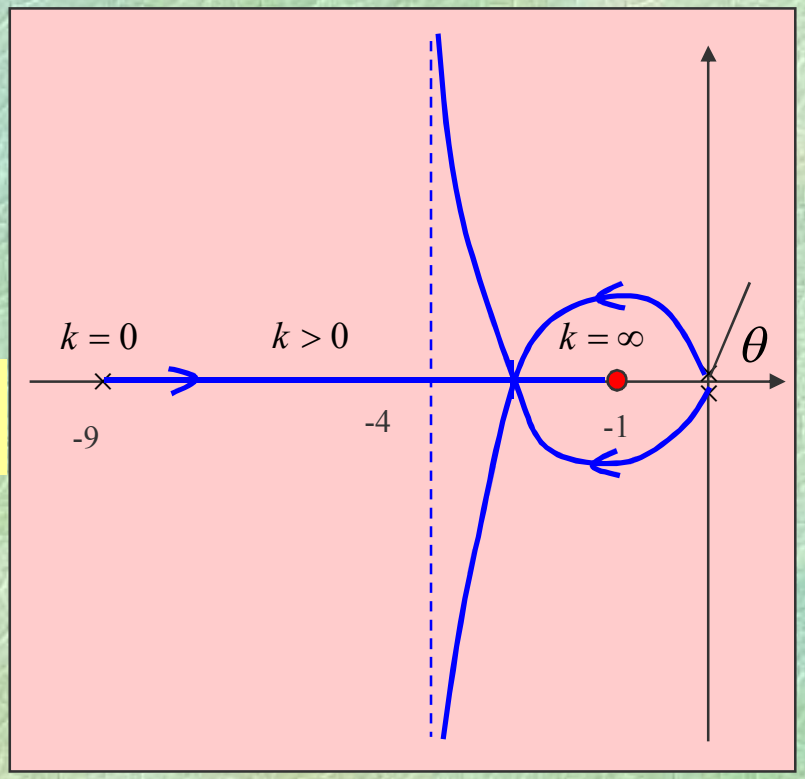
$$\delta=\frac{-9-(-1)}{2}=-4$$

$$Asymptotes\ \ are\ \frac{\pi}{2},\frac{3\pi}{2}$$

$$s=-\frac{(a+3)}{4}\pm\frac{1}{4}\sqrt{a^2-10a+9}=-\frac{(9+3)}{4}\pm\frac{1}{4}\sqrt{9^2-90+9}$$

$$s_1=-3\ \ s_2=-3$$

$$\theta=?\qquad \theta=90$$



Effect of moving poles

اثرات حرکت قطبها

Let **a=8**

$$1+\frac{k(s+1)}{s^2(s+8)}=0$$

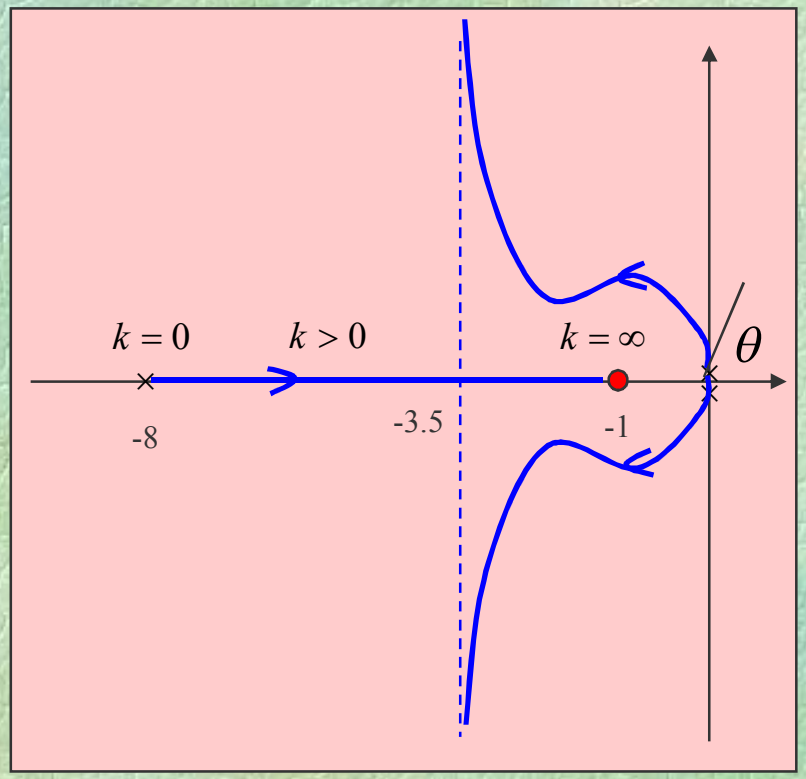
$$\delta=\frac{-8-(-1)}{2}=-3.5$$

$$Asymptotes\ \ are\ \ \frac{\pi}{2},\frac{3\pi}{2}$$

$$s=-\frac{(a+3)}{4}\pm\frac{1}{4}\sqrt{a^2-10a+9}=-\frac{(8+3)}{4}\pm\frac{1}{4}\sqrt{8^2-80+9}$$

$$no\ \ real\ \ s$$

$$\theta=?\qquad \theta=90$$



Effect of moving poles.

اثرات حرکت قطبها

Let **a=3**

$$1+\frac{k(s+1)}{s^2(s+3)}=0$$

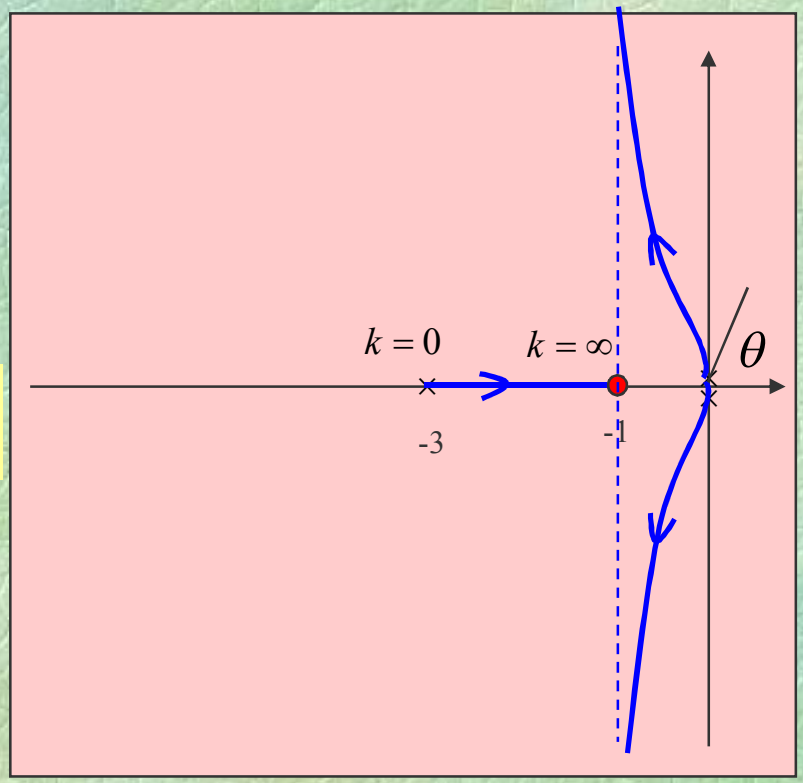
$$\delta=\frac{-3-(-1)}{2}=-1$$

$$Asymptotes\ are\ \frac{\pi}{2},\frac{3\pi}{2}$$

$$s=-\frac{(a+3)}{4}\pm\frac{1}{4}\sqrt{a^2-10a+9}=-\frac{(3+3)}{4}\pm\frac{1}{4}\sqrt{3^2-30+9}$$

$$no\ real\ s$$

$$\theta=?\qquad\theta=90$$



Effect of moving poles.

اثرات حرکت قطبها

Let **a=1**

$$1+\frac{k(s+1)}{s^2(s+1)}=0$$

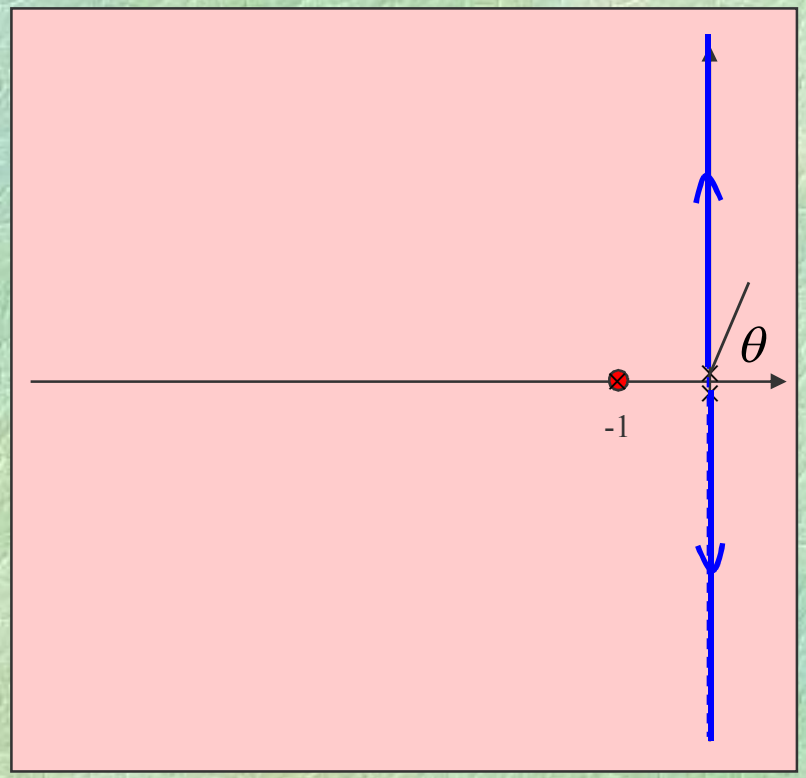
$$\delta=\frac{-1-(-1)}{2}=0$$

$$Asymptotes\ \ are\ \frac{\pi}{2},\frac{3\pi}{2}$$

$$s=-\frac{(a+3)}{4}\pm\frac{1}{4}\sqrt{a^2-10a+9}=-\frac{(3+3)}{4}\pm\frac{1}{4}\sqrt{1^2-10+9}$$

$$no\ \ real\ \ s$$

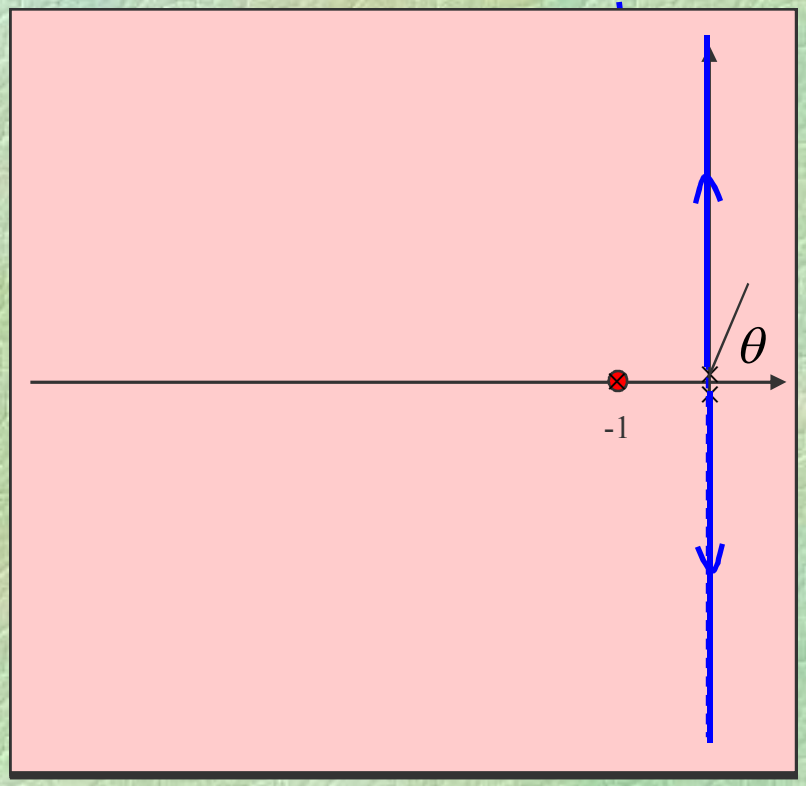
$$\theta=?\qquad \theta=90$$



Effect of moving poles.

اثرات حرکت قطبها

a= 1



Root contour (Multiple parameter variation)

کانتور ریشه ها (تغییرات چند پارامتر)

Remember root loci $1+kf(s)=0$

Suppose:

$$Q(s)+k_1P_1(s)+k_2P_2(s)=0$$

k_1 and k_2 are parameters

Step 1: Put one variable equal to zero. Let us k_2=0

$$Q(s)+k_1P_1(s)=0$$

$$1+k_1\frac{P_1(s)}{Q(s)}=0$$

$$f_1(s)$$

$$1+k_1f_1(s)=0$$

Step 2: Restore the value of k_2

$$Q(s)+k_1P_1(s)+k_2P_2(s)=0$$

$$1+k_2\frac{P_2(s)}{Q(s)+k_1P_1(s)}=0$$

$$f_2(s)$$

$$1+k_2f_2(s)=0$$

Example 2: Draw the root contour for following system.

$$s^3 + k_2 s^2 + k_1 s + k_1 = 0$$

مثال ۲: کانتور ریشه ها را برای سیستم زیر بیابید.

Step 1: Let k_2=0 then

$$1 + k_1 \frac{s+1}{s^3} = 0$$

Asymptotes are:

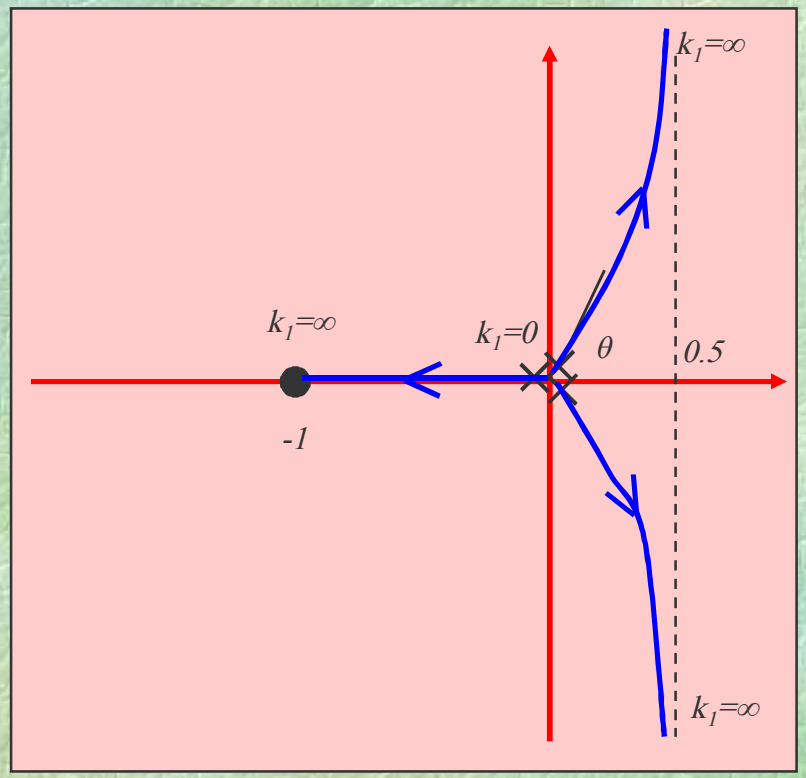
$$\frac{(2k+1)\pi}{3-1} = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\sigma = \frac{0+0+0-(-1)}{3-1} = \frac{1}{2}$$

Angles of departure is:

$$0 - (\theta + \theta + \theta) = \pm\pi$$

$$\theta = \pm\pi / 3 = \pm 60^\circ$$



Example 2: Draw the root contour for following system.

$$s^3 + k_2 s^2 + k_1 s + k_1 = 0$$

مثال ۲: کانتور ریشه ها را برای سیستم زیر بیابید.

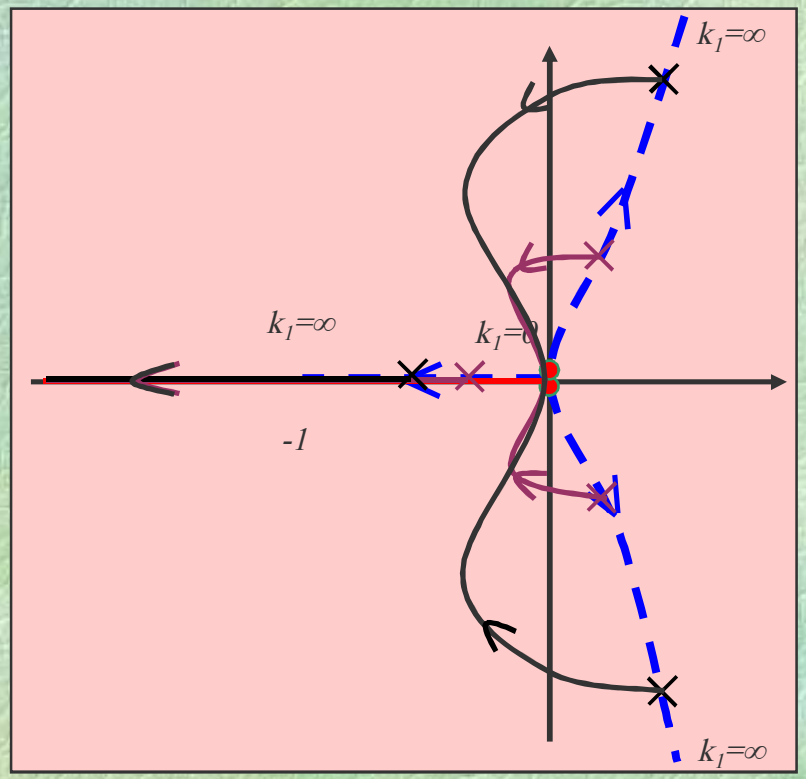
Step 2: Restore the value of k_2

$$1 + k_2 \frac{s^2}{s^3 + k_1 s + k_1} = 0$$

Let: k_1=0

Let: k_1=1

Let: k_1=10



Exercises

تمرینها

1 A unity feedback (negative sign) control system has an open loop transfer function

$$G(s) = \frac{k}{s(1+0.02s)(1+0.05s)}$$

Sketch the complete root loci, and find the corresponding k when the root loci crosses jw axis.

2 The open loop transfer function of a unity-feedback (negative sign) system with PD controller is:

$$G(s) = \frac{10(K_p + K_d s)}{s^2}$$

Sketch the root loci for different values of K_p and K_d. (Let *Kp=0,1,5,10.)*