LINEAR CONTROL SYSTEMS

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Lecture 20

Frequency domain charts

Topics to be covered include:

- Relative stability measures for minimum phase systems.
  - Gain margin.
  - Phase margin.
- Nichols chart or gain phase plot.
  - Stability analysis with gain phase plot.
- Bode plot.
  - Stability analysis with Bode plot.
- Step by step Bode plot construction.

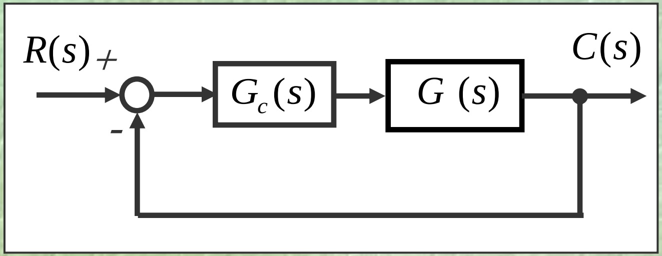
Stability margins

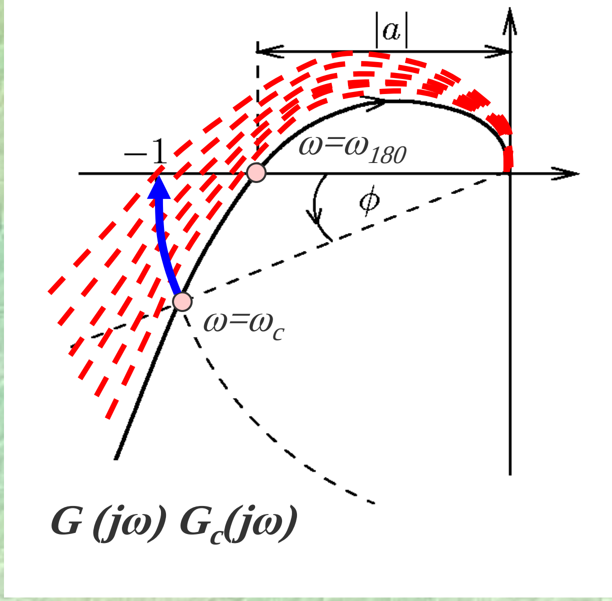
Stability Is Not A Yes/No Proposition

It's not enough to know that a system is stable or unstable. If a system is just barely stable, then a small gain in a system parameter could push the system over the edge, and you will often want to design systems with some margin of error.

If you're going to do that, you'll need some measure of how stable a system is. To get such measures - and there are at least two that are widely used.

Stability margins





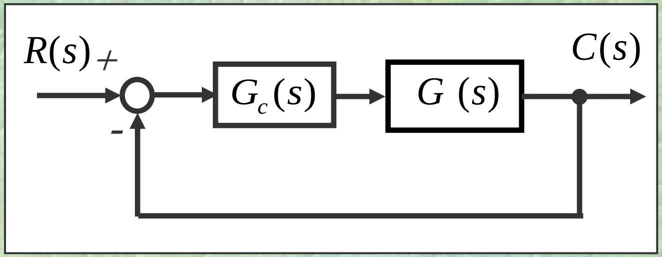
Phase Margin

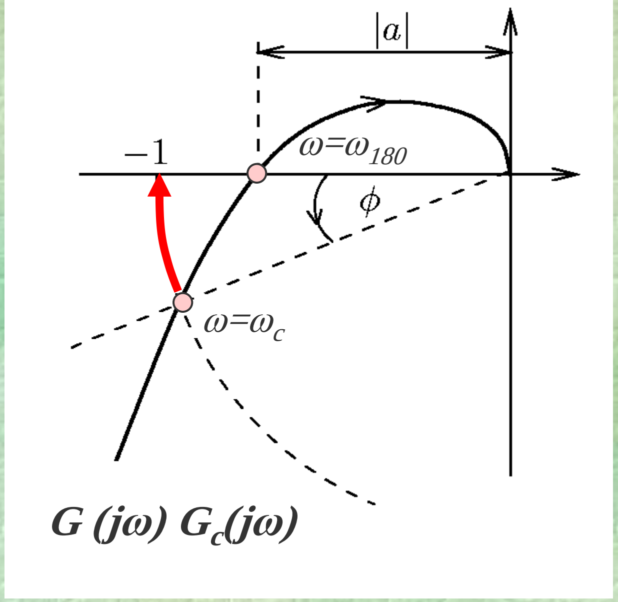
Phase margin is the **most** widely used measure of relative stability when working in the frequency domain.

We define phase margin as the phase (angle) that the frequency response would have to change to move to the -1 point.

ω_c : Is the gain crossover frequency

Phase margin computation





The phase margin, φ_m is defined as follows:

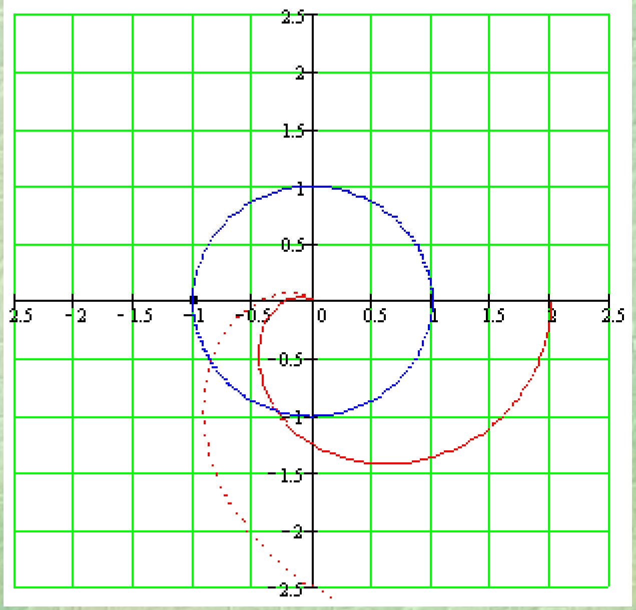
$$\varphi_m = 180 + \angle G(j\omega_c)G_c(j\omega_c)$$

How to derive ω_c

$$Let\ \ |G(j\omega)G_c(j\omega)| = 1 \qquad \Rightarrow \quad \omega_c = \surd$$

Stability margins

Phase Margin



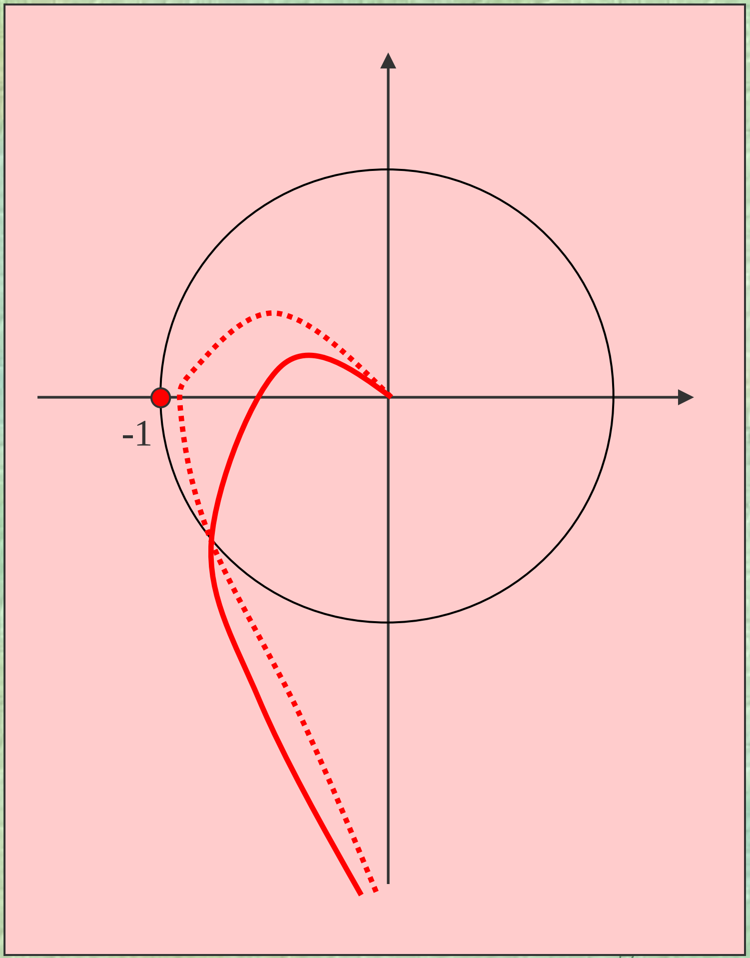
The two different Nyquist plot above would lead to two different phase margins. The system with the frequency response with the dashed line is less stable.

Stability margins

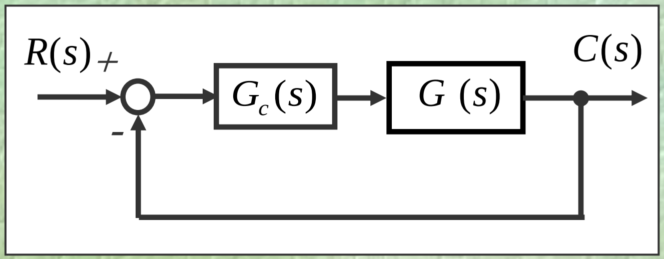
Phase Margin

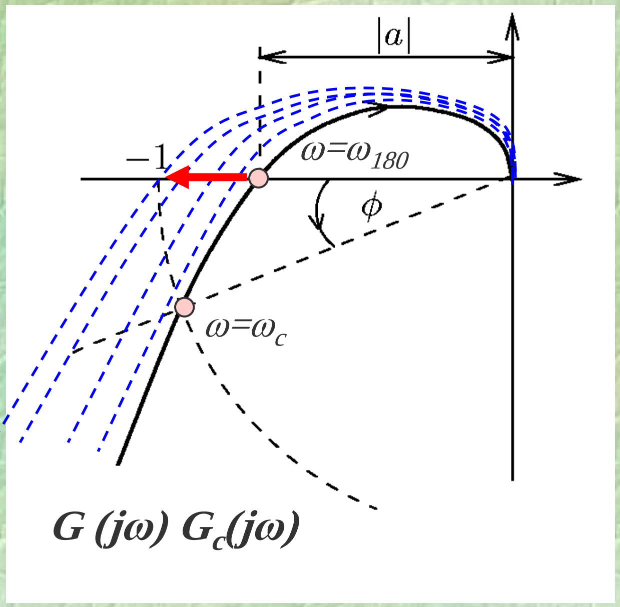
The two different gains shown for the Nyquist plot below would lead to the same phase margins. But do they have the same stability margin? Clearly no.

Thus we need another measure of relative stability.



Stability margins





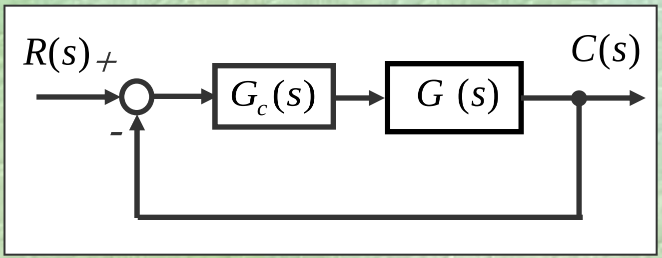
Gain Margin

Gain margin is **another** widely used measure of relative stability when working in the frequency domain.

We define gain margin as the gain that the frequency response would have to increase to move to the -1 point.

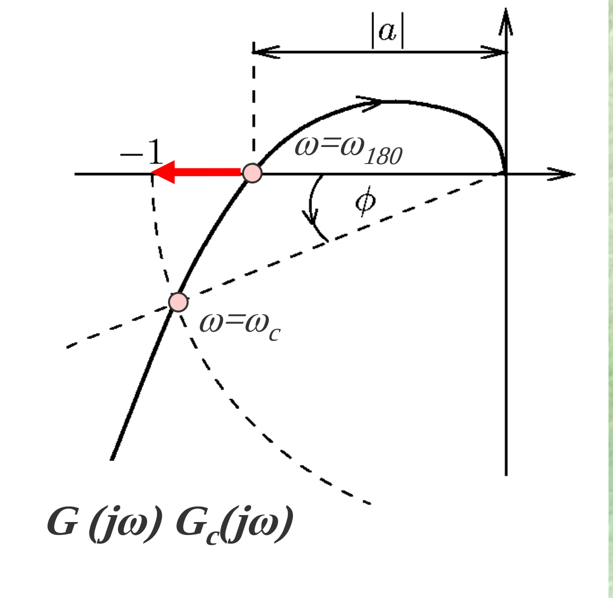
ω_{180} : Is the phase crossover frequency

Gain margin computation



The gain margin, GM is defined as follows:

$$GM = 1/\left|G(j\omega_{180})G_c(j\omega_{180})\right|$$



But gain margin is usually specified in db.

$$GM = 20\log\left(1/\left|G(j\omega_{180})G_c(j\omega_{180})\right|\right)$$

How to derive ω_{180}

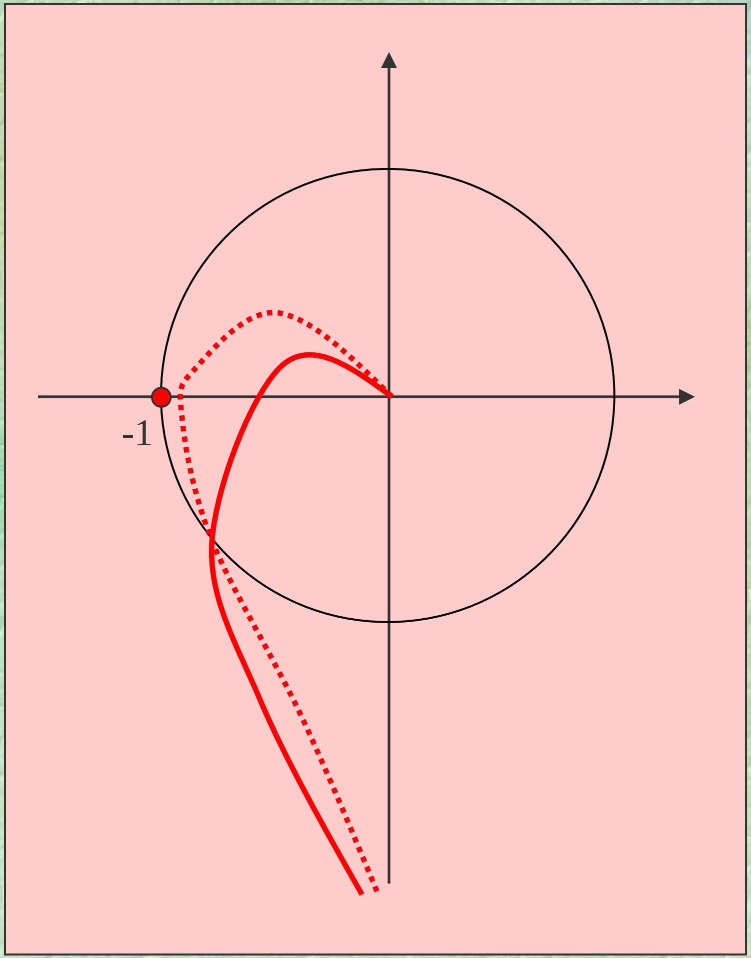
$$Let \;\; \angle\left(G(j\omega)G_c(j\omega)\right) = 180$$

$$or \;\; \mathrm{Im}\left(G(j\omega)G_c(j\omega)\right) = 0$$

$$\Rightarrow \quad \omega_{180} = \surd$$

Stability margins

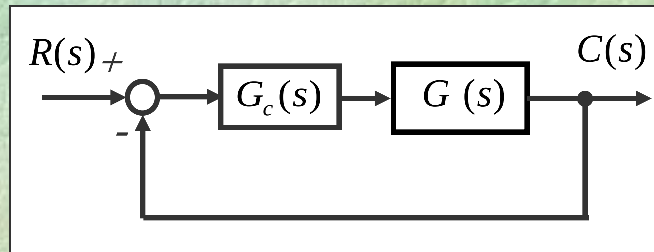
Phase and Gain Margin



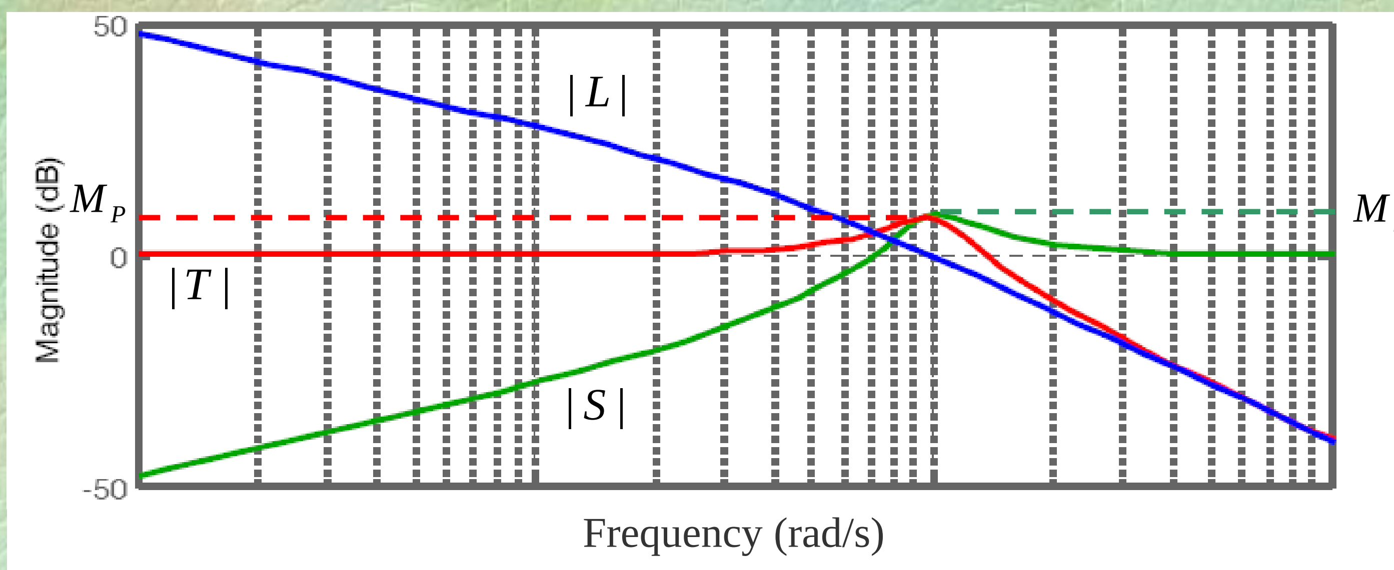
Same Phase Margin

But different Gain Margin

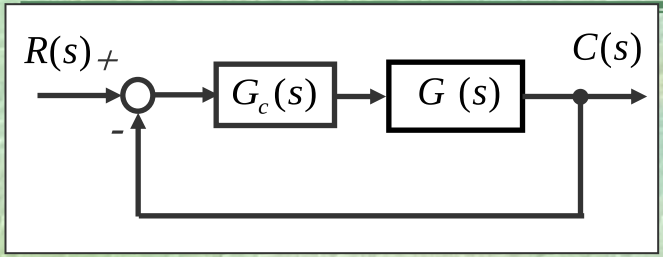
Sensitivity and complementary sensitivity peak



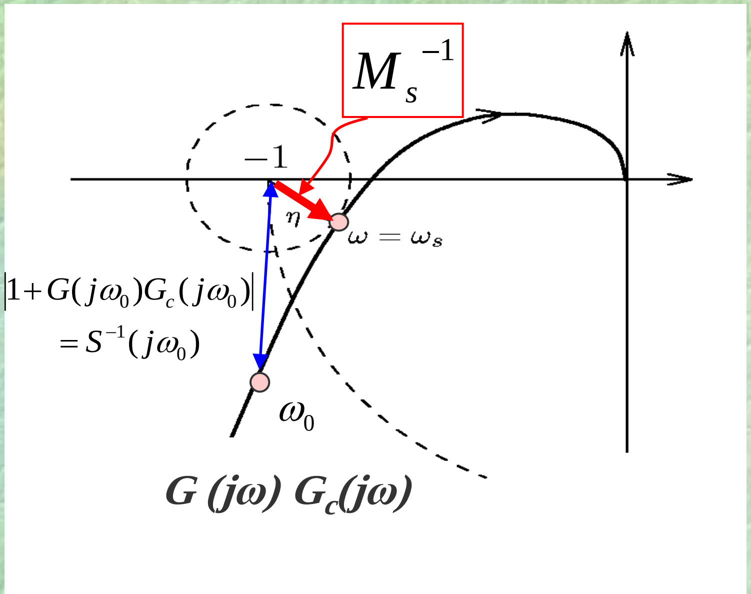
$$L(s) = G_c(s)G(s) \qquad T(s) = \frac{G_c(s)G(s)}{1+G_c(s)G(s)} = \frac{L(s)}{1+L(s)} \qquad S(s) = \frac{1}{1+G_c(s)G(s)} = \frac{1}{1+L(s)}$$



Sensitivity peak



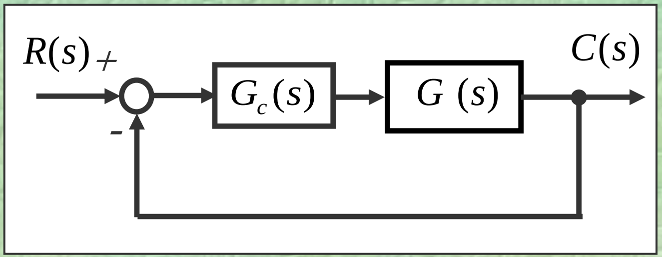
$$\left|S(j\omega)\right| = \frac{1}{\left|1+G(j\omega)G_c(j\omega)\right|}$$



The sensitivity peak, M_s is defined as follows:

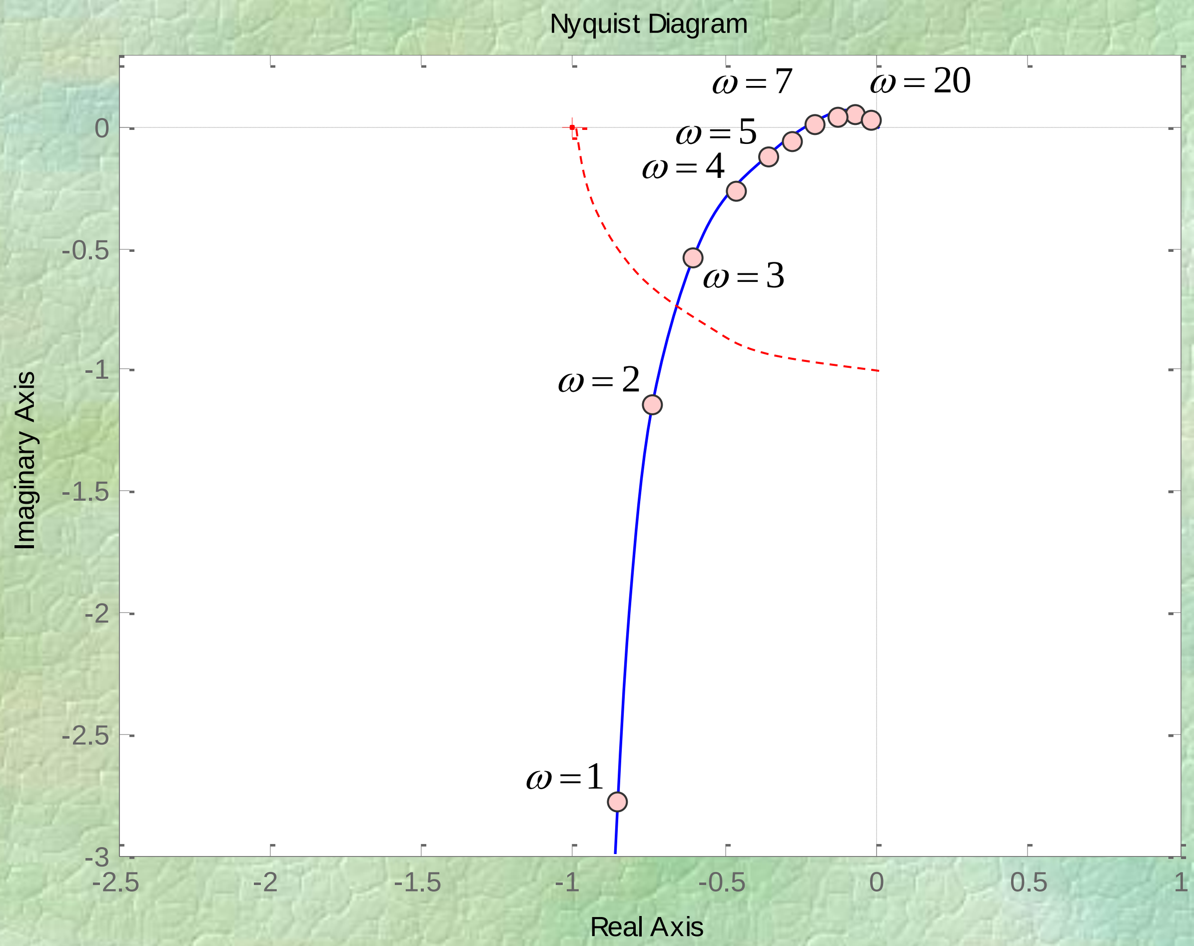
$$M_s = \max_{\omega}\left|S(j\omega)\right| = \frac{1}{\left|1+G(j\omega_s)G_c(j\omega_s)\right|}$$

Nyquist chart (polar plot) construction



Let $G_c(s)G(s) = \dfrac{150}{s(s+5)(s+10)}$

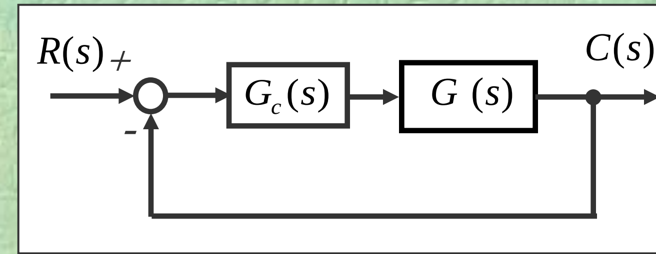
ω	$G_c(j\omega)G(j\omega)$
1	$2.93\angle -107^\circ$
2	$1.37\angle -123^\circ$
3	$0.82\angle -138^\circ$
4	$0.54\angle -151^\circ$
5	$0.38\angle -162^\circ$
6	$0.27\angle -171^\circ$
7	$0.20\angle -179^\circ$
8	$0.16\angle -187^\circ$
9	$0.12\angle -193^\circ$
20	$0.02\angle -229^\circ$



$G_c(j1)G(j1) = \dfrac{150}{j(j1+5)(j1+10)} = 2.93\angle -107$

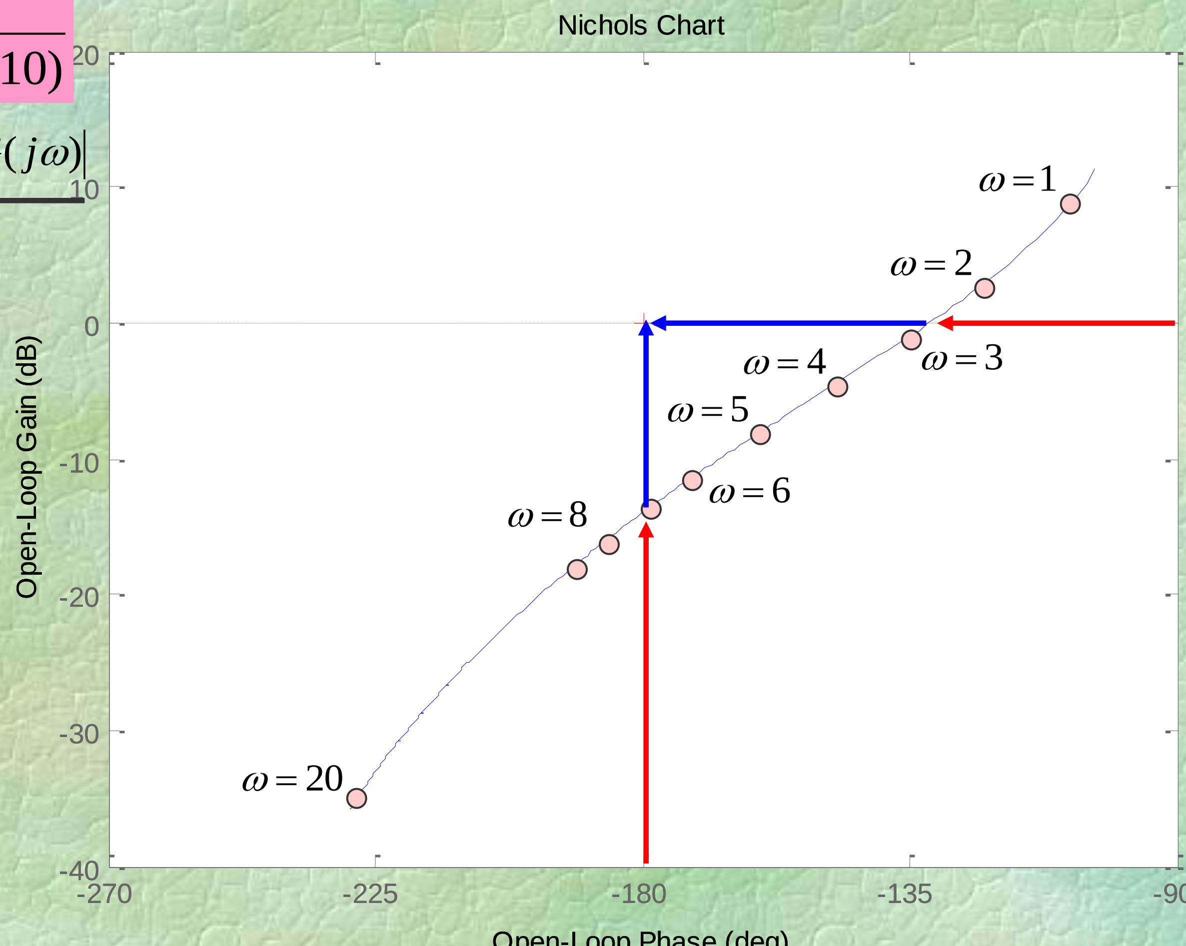
$\omega_c = 2.6$ $PM = 45^\circ$ $\omega_{180} = 7$ $GM = ???\ db$

Nichols chart (gain phase plot) construction



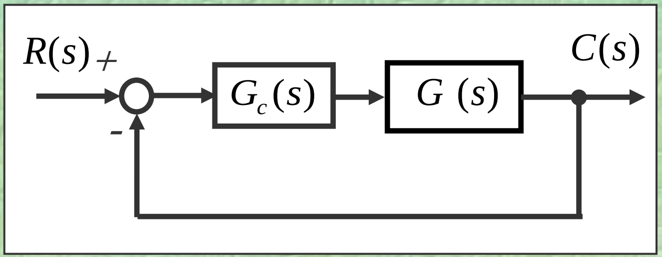
Let $G_c(s)G(s) = \dfrac{150}{s(s+5)(s+10)}$

ω	$G_c(j\omega)G(j\omega)$	$20\log\lvert G_c(j\omega)G(j\omega)\rvert$
1	$2.93\angle -107^\circ$	$9.33\ db$
2	$1.37\angle -123^\circ$	$2.71\ db$
3	$0.82\angle -138^\circ$	$-1.71\ db$
4	$0.54\angle -151^\circ$	$-5.29\ db$
5	$0.38\angle -162^\circ$	$-8.42\ db$
6	$0.27\angle -171^\circ$	$-11.23\ db$
7	$0.20\angle -179^\circ$	$-13.80\ db$
8	$0.16\angle -187^\circ$	$-16.18\ db$
9	$0.12\angle -193^\circ$	$-18.39\ db$
20	$0.02\angle -229^\circ$	$-35.77\ db$



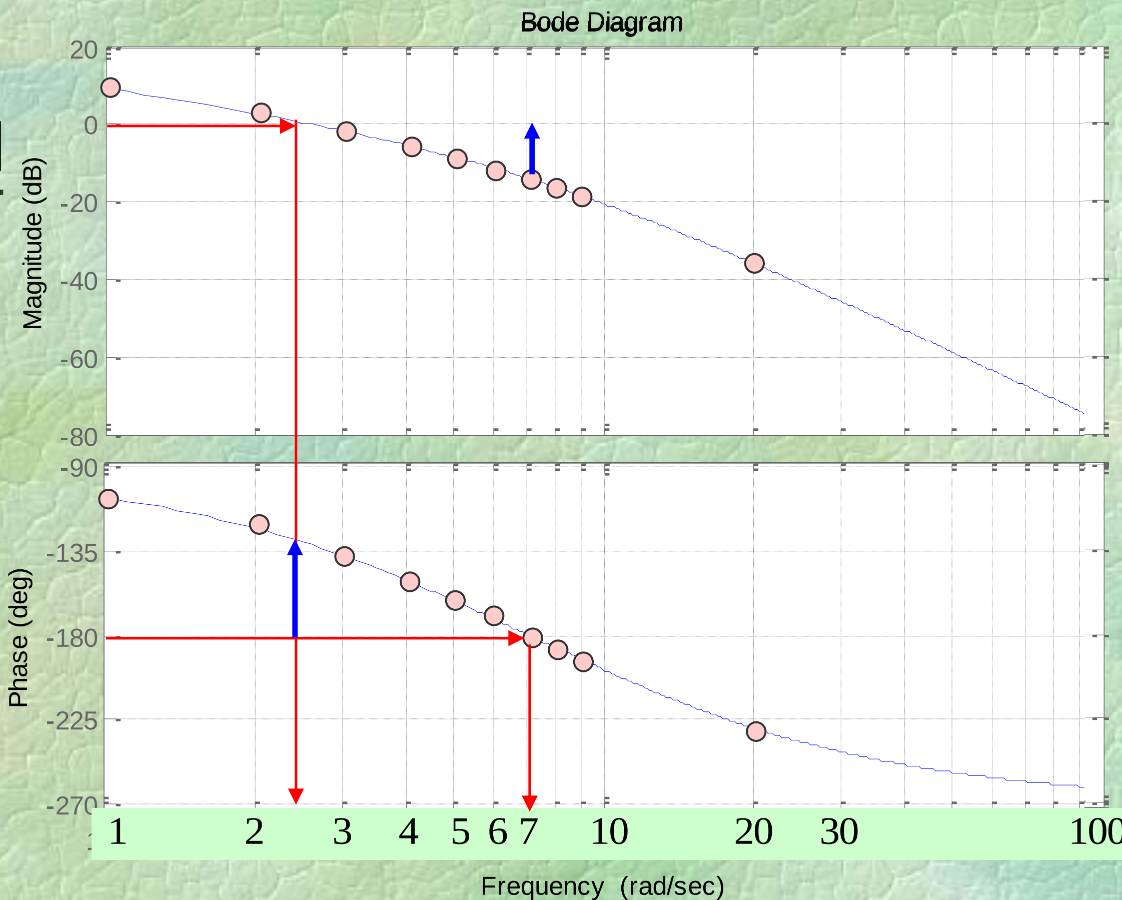
$\omega_c = 2.6$ $PM = 45^\circ$ $\omega_{180} = 7$ $GM = 13.8\ db$

Bode plot construction



Let $G_c(s)G(s) = \dfrac{150}{s(s+5)(s+10)}$

ω	$G_c(j\omega)G(j\omega)$	$20\log\lvert G_c(j\omega)G(j\omega)\rvert$
1	$2.93\angle -107^\circ$	$9.33\ db$
2	$1.37\angle -123^\circ$	$2.71\ db$
3	$0.82\angle -138^\circ$	$-1.71\ db$
4	$0.54\angle -151^\circ$	$-5.29\ db$
5	$0.38\angle -162^\circ$	$-8.42\ db$
6	$0.27\angle -171^\circ$	$-11.23\ db$
7	$0.20\angle -179^\circ$	$-13.80\ db$
8	$0.16\angle -187^\circ$	$-16.18\ db$
9	$0.12\angle -193^\circ$	$-18.39\ db$
20	$0.02\angle -229^\circ$	$-35.77\ db$



$\omega_c = 2.5$ | $PM = 48^\circ$ | $\omega_{180} = 7$ | $GM = 13.8\ db$

Step by step bode plot construction

$$\text{Let} \quad G_c(s)G(s) = \frac{\bar{k}(s+z_1)(s+z_2)}{s^2(s+p_1)(s+p_2)(s+p_3)}$$

Step 1:

$$G_c(s)G(s) = \frac{\bar{k}z_1z_2}{p_1p_2p_3}\frac{(s/z_1+1)(s/z_2+1)}{s^2(s/p_1+1)(s/p_2+1)(s/p_3+1)} = k\frac{(s/z_1+1)(s/z_2+1)}{s^2(s/p_1+1)(s/p_2+1)(s/p_3+1)}$$

Step 2:

$$20\log\left|G_c(j\omega)G(j\omega)\right| = 20\log|k| + 20\log|j\omega/z_1+1| + 20\log|j\omega/z_2+1|$$
$$+20\log\left|1/(j\omega)^2\right| + 20\log\left|\frac{1}{j\omega/p_1+1}\right| + 20\log\left|\frac{1}{j\omega/p_2+1}\right| + 20\log\left|\frac{1}{j\omega/p_3+1}\right|$$

Step 3:

$$\angle G_c(j\omega)G(j\omega) = \angle k + \angle(j\omega/z_1+1) + \angle(j\omega/z_2+1) + \angle(1/(j\omega)^2)$$
$$+\angle\left(\frac{1}{j\omega/p_1+1}\right) + \angle\left(\frac{1}{j\omega/p_2+1}\right) + \angle\left(\frac{1}{j\omega/p_3+1}\right)$$

Step by step bode plot construction

$$\text{Let}\quad G_c(s)G(s) = k\,\frac{(s/z_1+1)(s/z_2+1)}{s^2(s/p_1+1)(s/p_2+1)(s/p_3+1)}$$

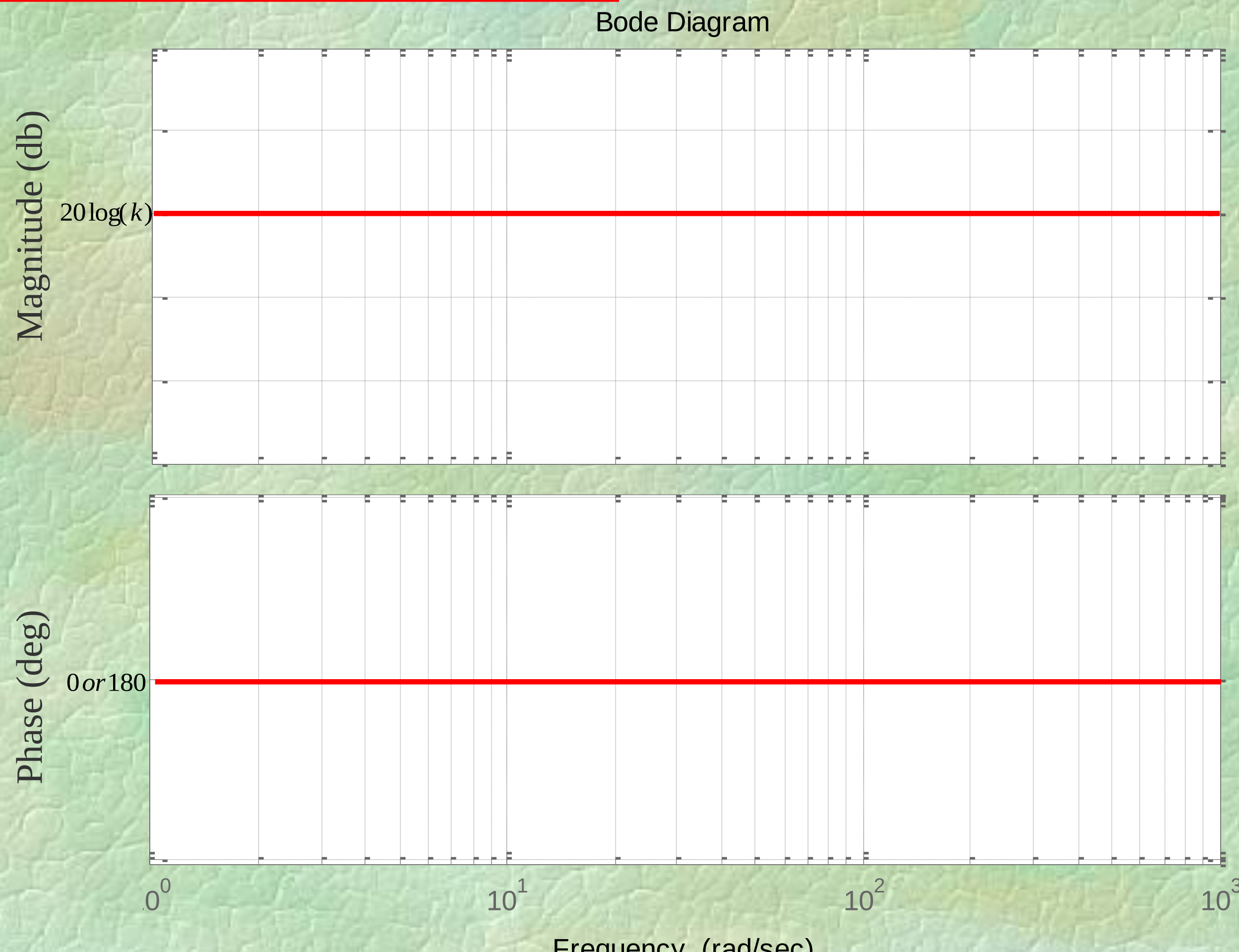
$$20\log\left|G_c(j\omega)G(j\omega)\right| = 20\log|k| + 20\log\left|j\omega/z_1+1\right| + 20\log\left|j\omega/z_2+1\right| + 20\log\left|1/(j\omega)^2\right|$$
$$+20\log\left|\frac{1}{j\omega/p_1+1}\right| + 20\log\left|\frac{1}{j\omega/p_2+1}\right| + 20\log\left|\frac{1}{j\omega/p_3+1}\right|$$

$$\angle G_c(j\omega)G(j\omega) = \angle k + \angle(j\omega/z_1+1) + \angle(j\omega/z_2+1) + \angle(1/(j\omega)^2)$$
$$+\angle\left(\frac{1}{j\omega/p_1+1}\right) + \angle\left(\frac{1}{j\omega/p_2+1}\right) + \angle\left(\frac{1}{j\omega/p_3+1}\right)$$

Step by step bode plot construction

$$20\log\left|G_c(j\omega)G(j\omega)\right| = 20\log\left|k\right|$$

$$\angle G_c(j\omega)G(j\omega) = \angle k$$



Why 0 or 180?

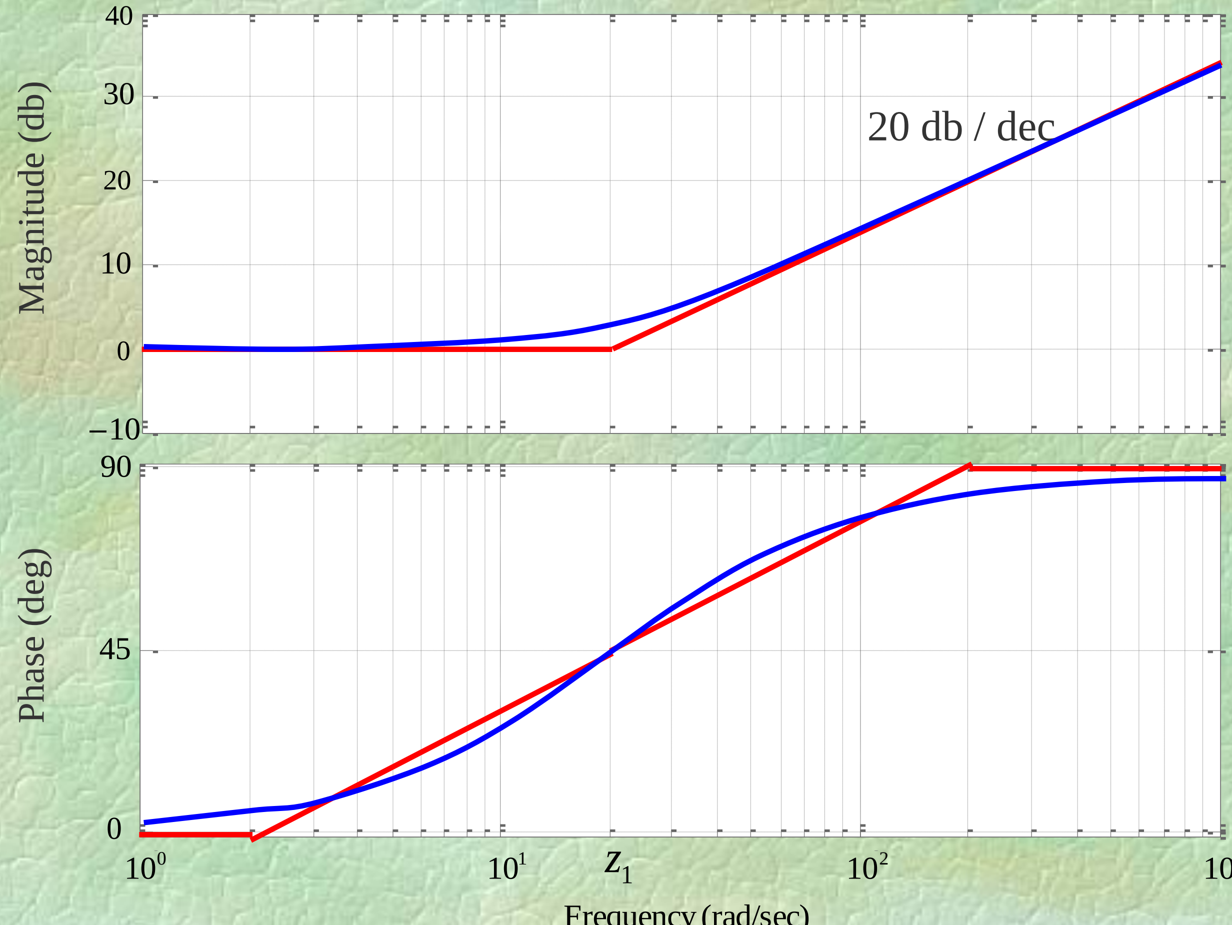
It depends to sign of k.

Step by step bode plot construction

$$20\log|G_c(j\omega)G(j\omega)| = 20\log|j\omega/z_1+1|$$

$$\angle G_c(j\omega)G(j\omega) = \angle(j\omega/z_1+1)$$

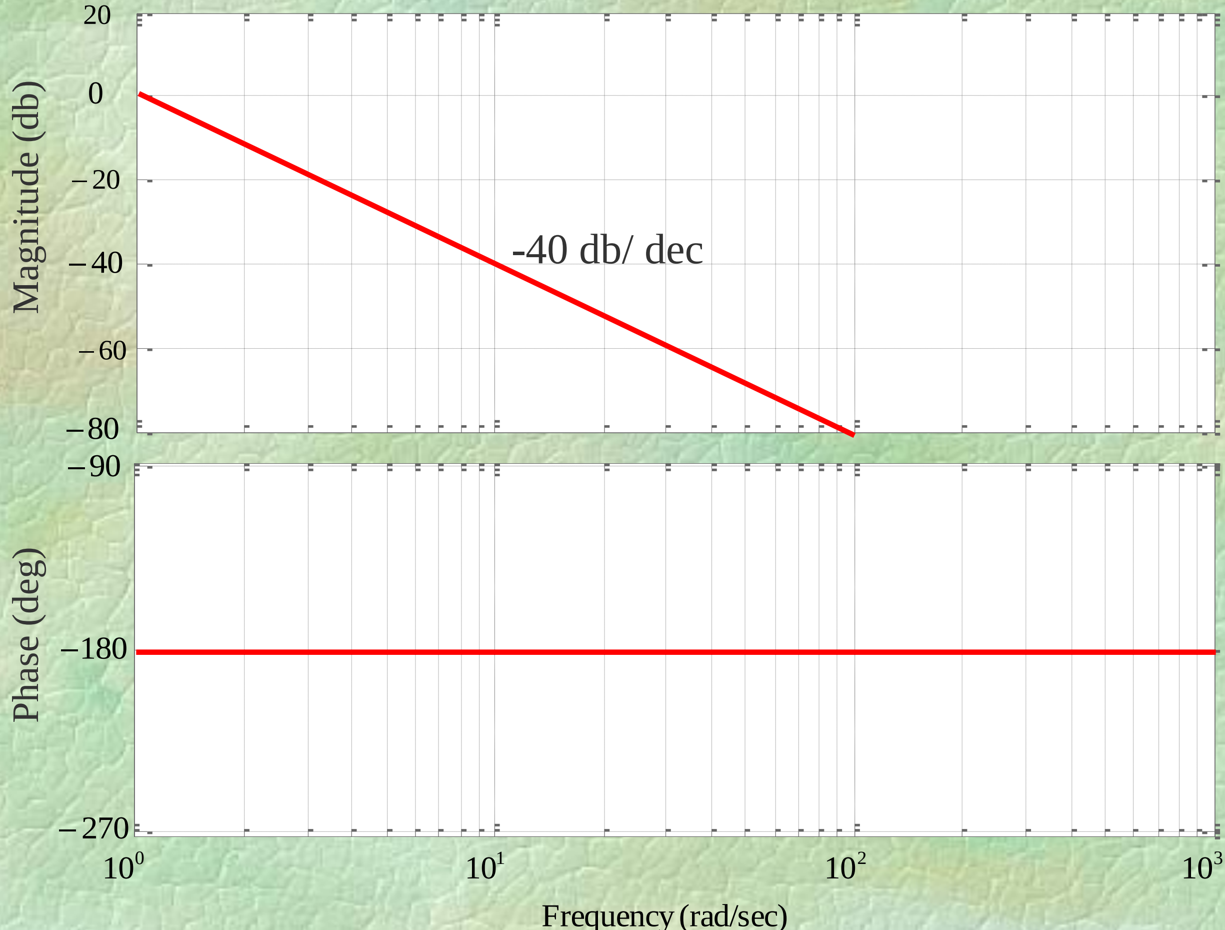
Let $z_1 = 20$



Step by step bode plot construction

$$20\log\left|G_c(j\omega)G(j\omega)\right| = 20\log\left|1/(j\omega)^2\right|$$

$$\angle G_c(j\omega)G(j\omega) = \angle(1/(j\omega)^2)$$

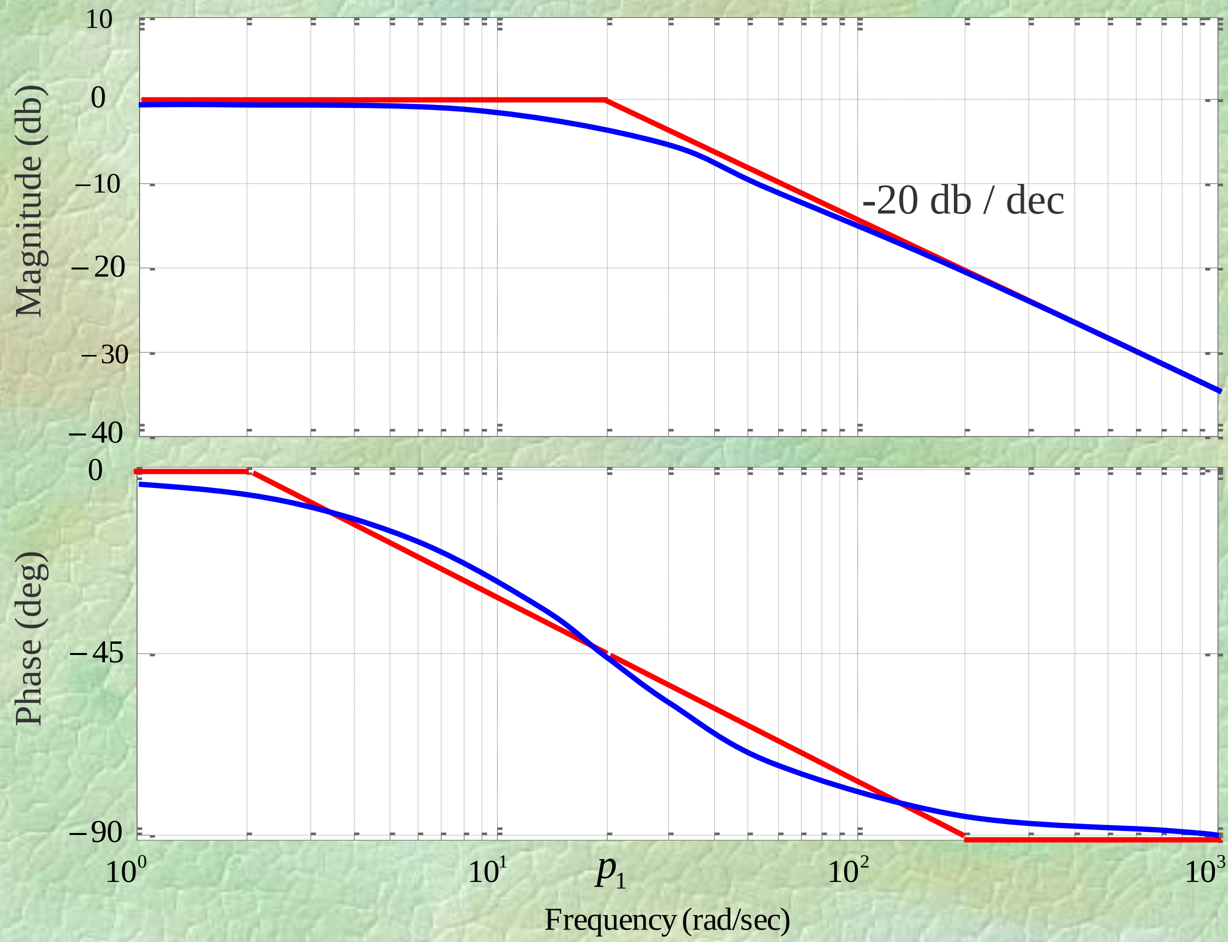


Step by step bode plot construction

$$20\log\left|G_c(j\omega)G(j\omega)\right| = 20\log\left|\frac{1}{j\omega/p_1+1}\right|$$

$$\angle G_c(j\omega)G(j\omega) = \angle\left(\frac{1}{j\omega/p_1+1}\right)$$

Let $p_1 = 20$



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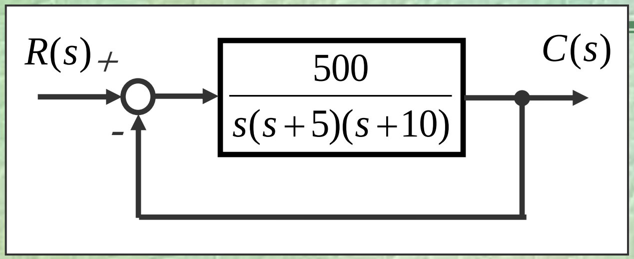
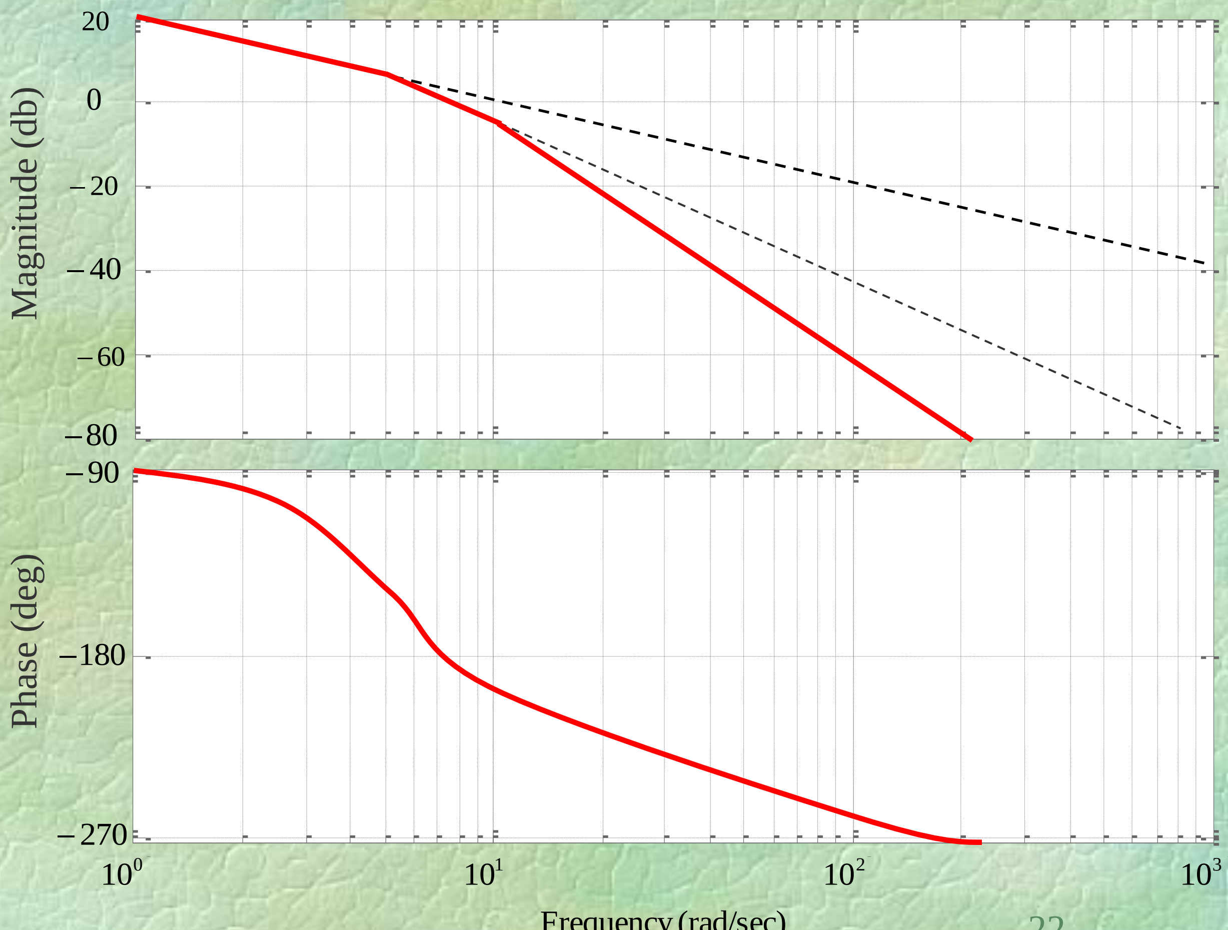
Example 1: Derive the GM and PM of following system by use of Bode plot.



Open loop transfer function is:

$$G(s)=\frac{500}{s(s+5)(s+10)}$$

$$=\frac{10}{s(s/5+1)(s/10+1)}$$



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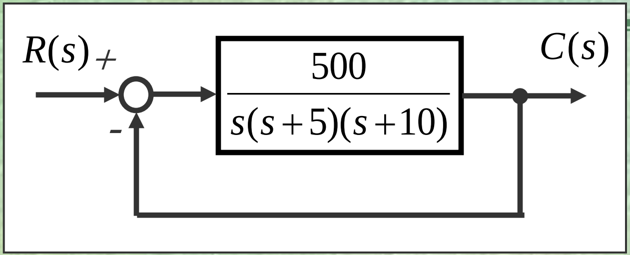
Example 1: Derive the GM and PM of following system by use of Bode plot.

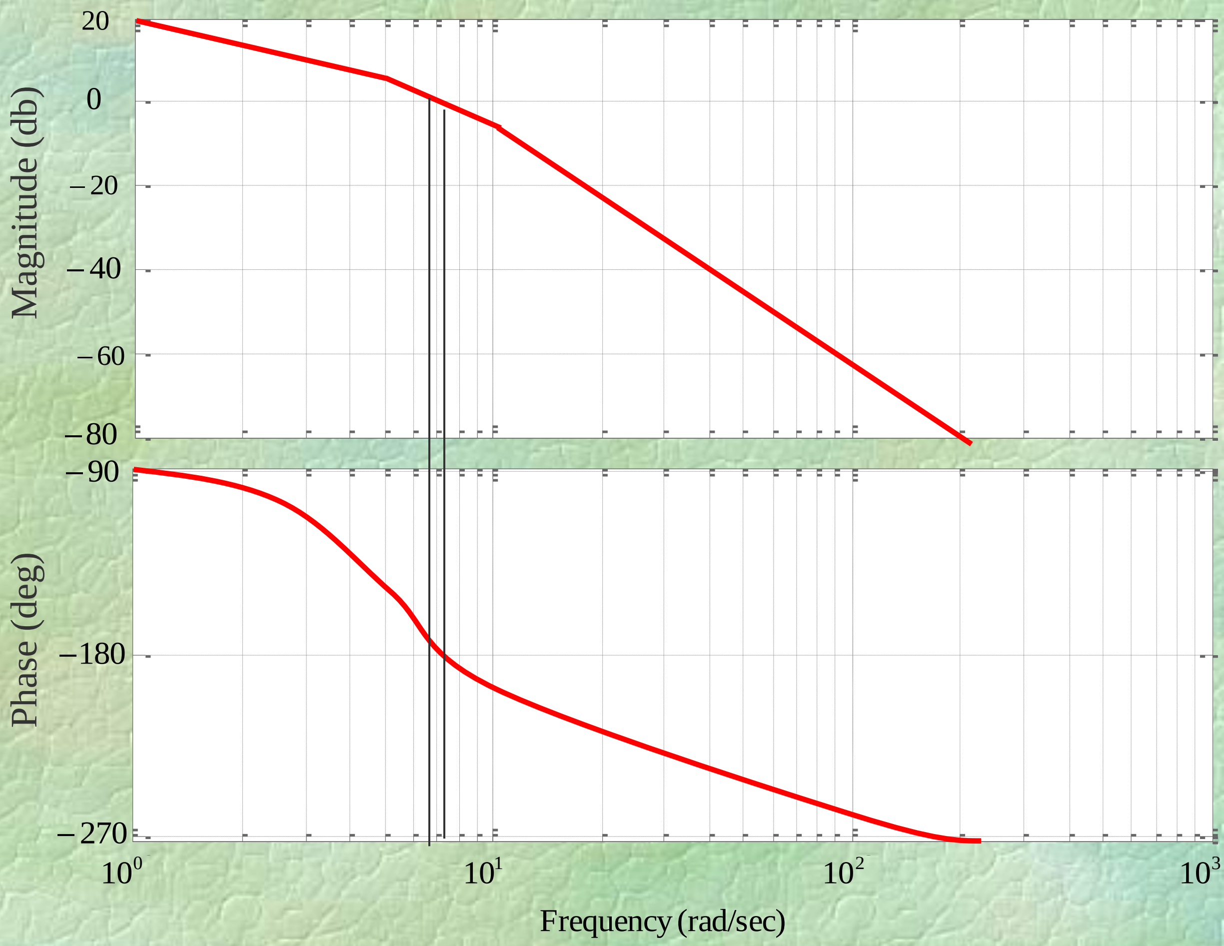


$$\omega_c = 6.5\ rad/\sec$$

$$\Rightarrow \quad PM = 10^\circ$$

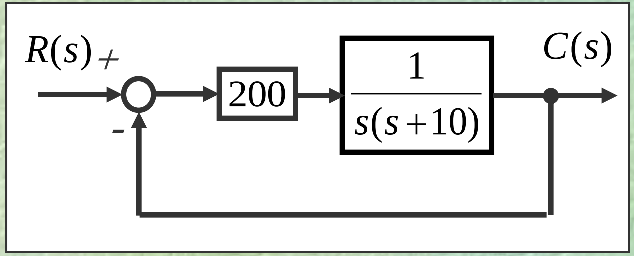
$$\omega_{180} = 7.2\ rad/\sec$$

$$\Rightarrow \quad GM = 3db$$



Exercises

1: Derive the gain crossover frequency, phase crossover frequency , GM and PM of following system by use of Bode plot.



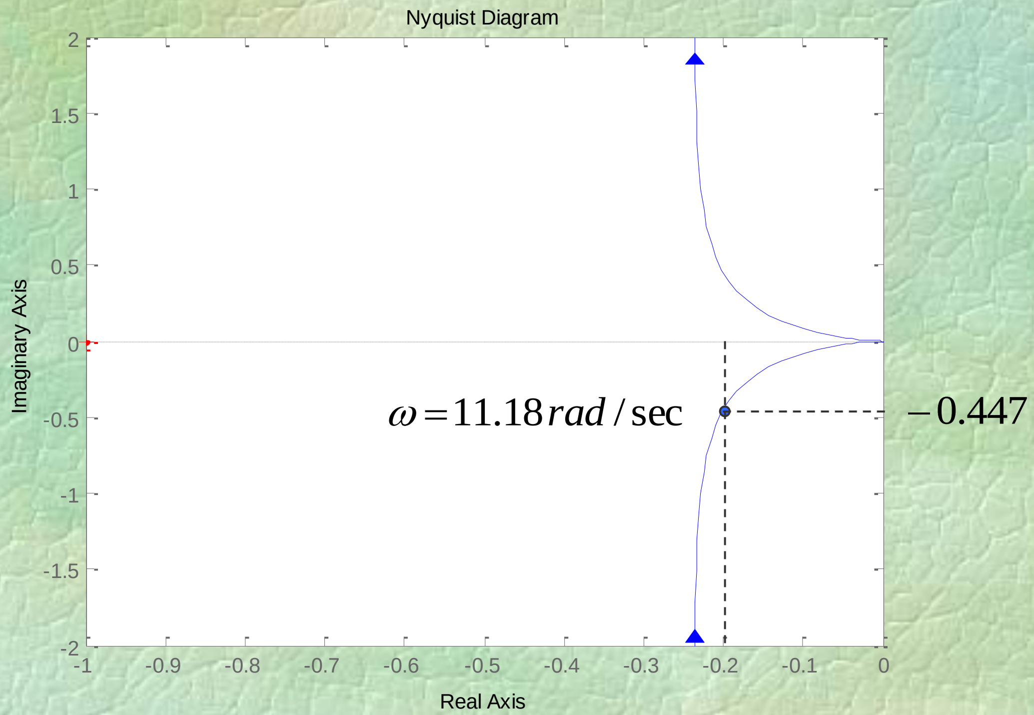
$Answer: \omega_c = 12.5\ ,\ \omega_{180} = \infty, GM = \infty\ and\ \varphi_m = 38^\circ$

Exercises

2 The polar plot of an open loop system with negative unit feedback is shown.

a) Find the open loop transfer function.

b) Find the closed loop transfer function.



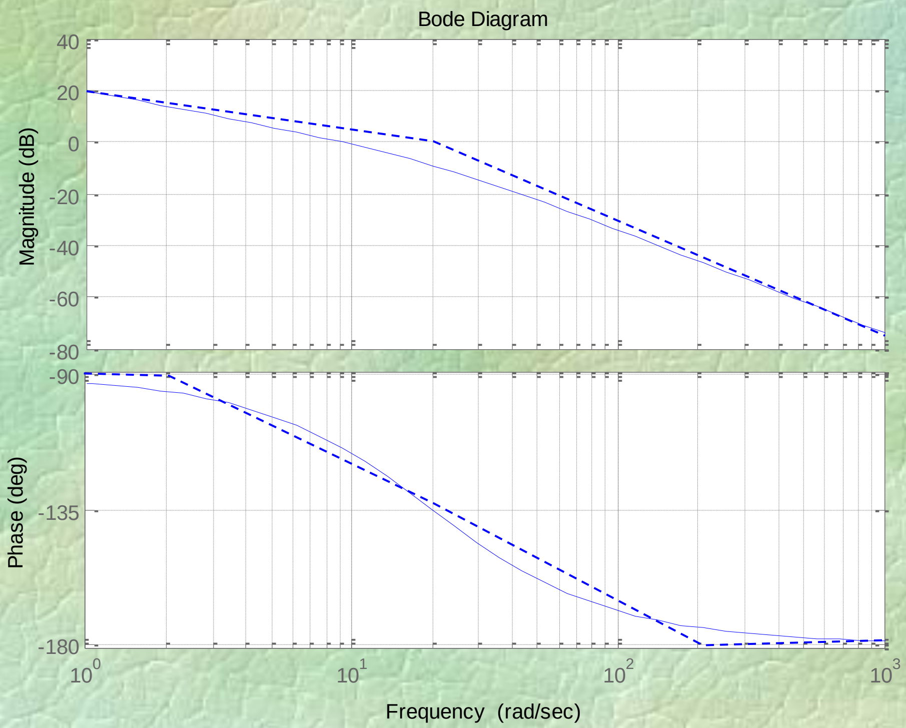
$$answer\ a: \frac{150}{s(s+25)} \quad b: \frac{150}{s^2+25s+150}$$

Exercises

3 The Bode plot of an open loop system with negative unit feedback is shown.

a) Find the open loop transfer function.

b) Find the closed loop transfer function.



$$answer\ a: \frac{200}{s(s+20)} \quad b: \frac{200}{s^2+20s+200}$$