LINEAR CONTROL SYSTEMS

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Lecture 3

Different representations of control systems

Topics to be covered include:

- High Order Differential Equation. (HODE model)
- State Space model. (SS model)
- Transfer Function. (TF model)
- State Diagram. (SD model)

High order differential equation (HODE).

معادلات دیفرانسیل مرتبه بالا

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \ldots\ldots + a_1\frac{dy}{dt} + a_0 y = \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \ldots\ldots + b_1\frac{du}{dt} + b_0 u$$

y is the output

u is the input

HODE model

Example 1: A high order differential equation (HODE).

مثال ۱: یک معادله دیفرانسیل مرتبه بالا

خروجی

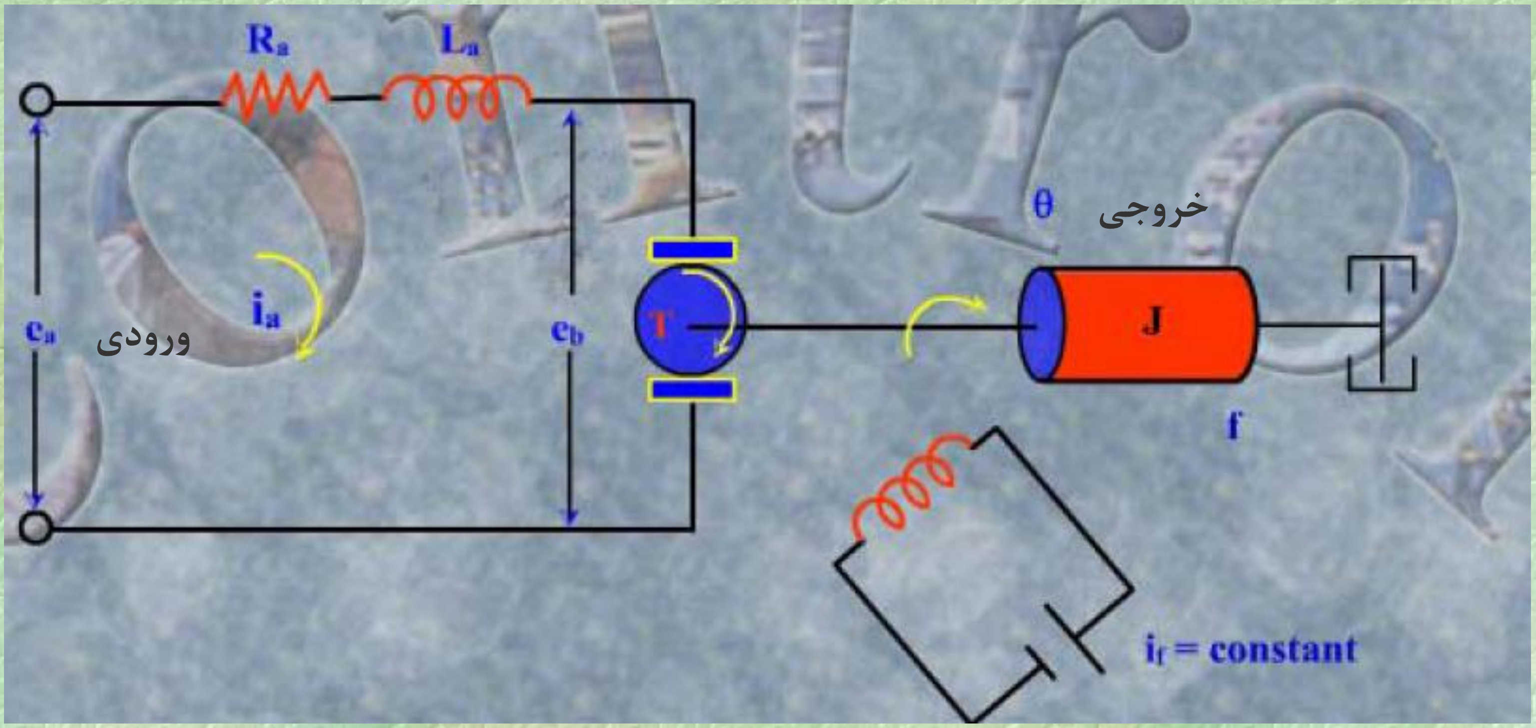
ورودی

$$u - b\dot{x} = M\ddot{x} \;\rightarrow\; \ddot{x} + \frac{b}{M}\dot{x} = \frac{1}{M}u \quad , \; with\; a = \ddot{x}$$

HODE model

Example 2: Another high order differential equation.

مثال۲: مثالی دیگر از معادله دیفرانسیل مرتبه بالا



$$J\frac{d^2\theta}{dt^2}+f\frac{d\theta}{dt}=T$$

$$=Ki_a$$

$$L_a\frac{di_a}{dt}+R_a i_a+e_b=e_a$$

$$e_b=K_b\frac{d\theta}{dt}$$

HODE model

State Space Models (SS)

معادلات فضای حالت

For continuous time systems

$$\frac{dx}{dt} = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

SS model

For linear time invariant continuous time systems

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

SS model

State Space Models

معادلات فضای حالت

General form of LTI systems in state space form

فرم کلی فضای حالتی سیستمهای LTI

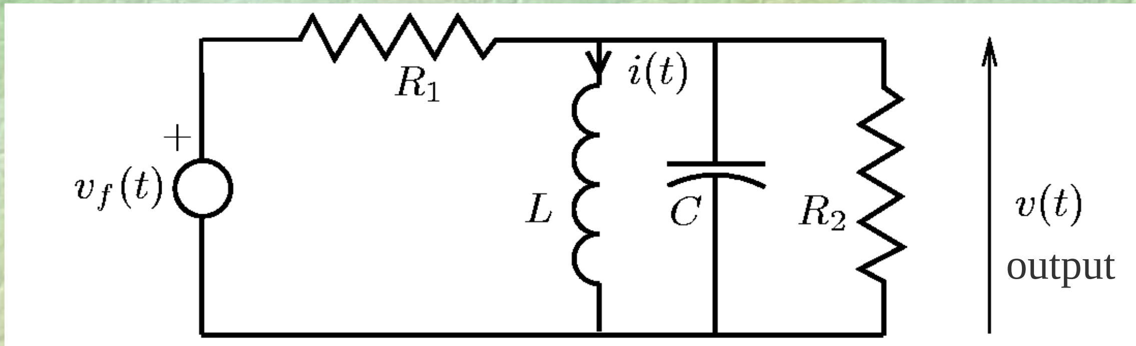
$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ . \\ . \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & . & . & a_{1n} \\ a_{21} & a_{22} & . & . & a_{2n} \\ . & . & . & . & . \\ . & . & . & . & . \\ a_{n1} & a_{n2} & . & . & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ x_n \end{bmatrix} + \begin{bmatrix} b_{11} & b_{12} & . & . & b_{1p} \\ b_{21} & b_{22} & . & . & b_{2p} \\ . & . & . & . & . \\ . & . & . & . & . \\ b_{n1} & b_{n2} & . & . & b_{np} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ . \\ . \\ u_p \end{bmatrix}$$

SS model

Example 3: A linear time invariant continuous time systems

مثال۳: یک سیستم خطی غیر متغیر با زمان (LTI)



$$v(t) = L\frac{di(t)}{dt}$$

$$\frac{v_f(t) - v(t)}{R_1} = i(t) + C\frac{dv(t)}{dt} + \frac{v(t)}{R_2}$$

SS model

Example 3: Continue

مثال۳: ادامه

The equations can be rearranged as follows: ساده سازی

$$\frac{di(t)}{dt} = \frac{1}{L} v(t)$$

$$\frac{dv(t)}{dt} = -\frac{1}{C} i(t) - \left(\frac{1}{R_1 C} + \frac{1}{R_2 C} \right) v(t) + \frac{1}{R_1 C} v_f(t)$$

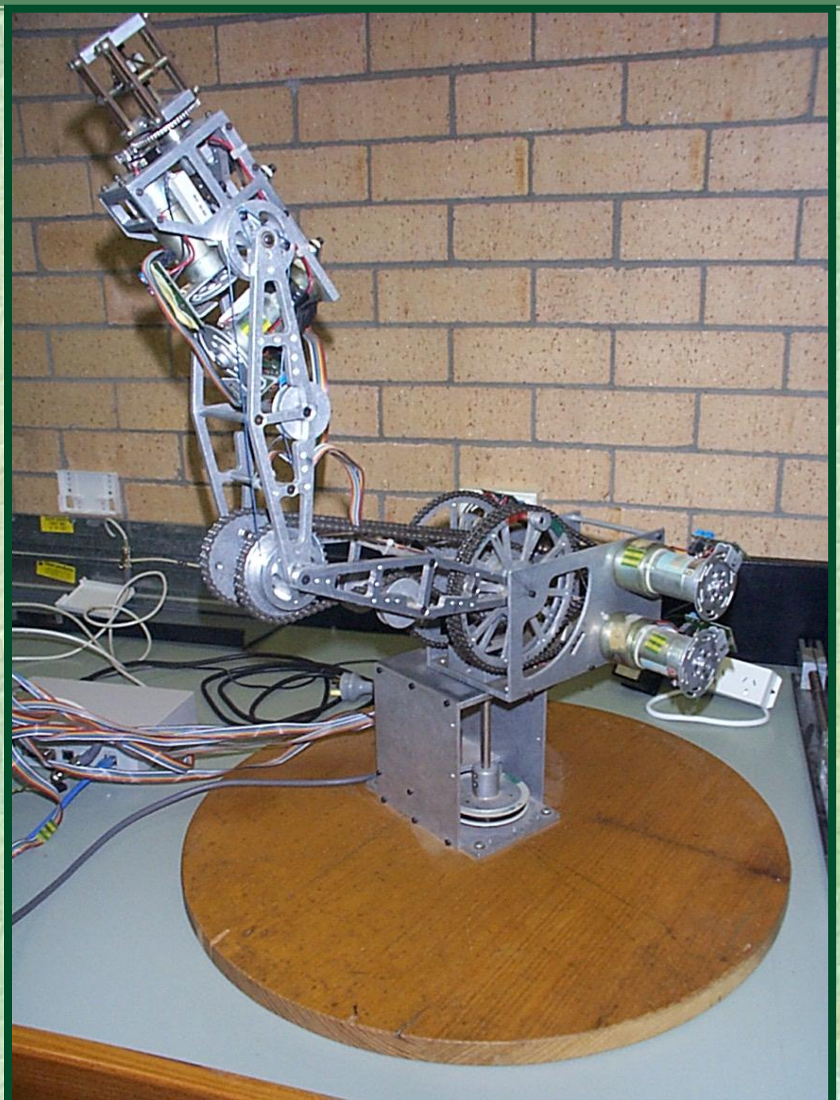
$$c(t) = v(t)$$

We have a linear state space model with مدل خطی فضای حالتی

$$\mathbf{A} = \begin{bmatrix} 0 & \frac{1}{L} \\ -\frac{1}{C} & -\left(\frac{1}{R_1 C} + \frac{1}{R_2 C}\right) \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{R_1 C} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 0 & 1 \end{bmatrix}; \quad \mathbf{D} = \mathbf{0}$$

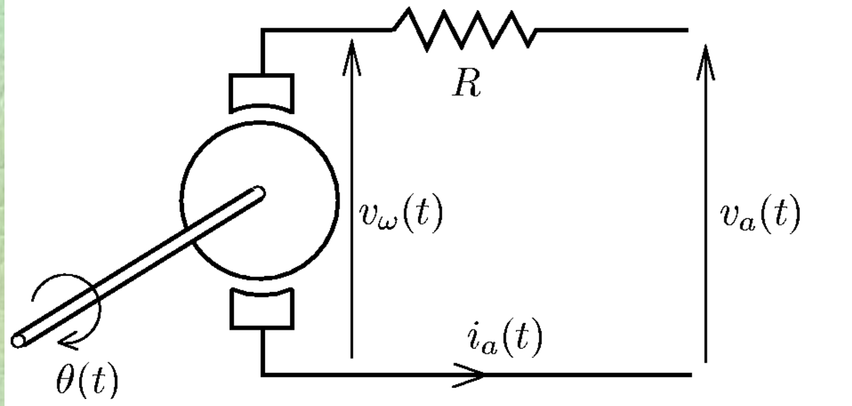
A demonstration robot containing several servo motors

مثالی از موتور dc در ربات



Example 4: DC motor

مثال ۴: موتور DC



J	- be the inertia of the shaft
$\tau_e(t)$	- the electrical torque
$i_a(t)$	- the armature current
$k_1; k_2$	- constants
R	- the armature resistance

$$J\ddot{\theta}(t) = \tau_e(t) = k_1 i_a(t)$$

$$v_\omega(t) = k_2\dot{\theta}(t)$$

$$i_a(t) = \frac{v_a(t) - k_2\dot{\theta}(t)}{R}$$

$$Let \quad x_1(t) = \theta(t) \quad And \quad x_2(t) = \dot{\theta}(t)$$

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & -\frac{k_1 k_2}{JR} \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{k_1}{JR} \end{bmatrix} v_a(t)$$

HODE model → SS model

Different representations

نمایشهای مختلف

HODE model → SS model

TF model

SD model

Linear high order differential equation

معادلات دیفرانسیل مرتبه بالای خطی

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1} y}{dt^{n-1}} + \dots\dots + a_1\frac{dy}{dt} + a_0 y = \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1} u}{dt^{m-1}} + \dots\dots + b_1\frac{du}{dt} + b_0 u$$

The study of differential equations of the type described above is a rich and interesting subject. Of all the methods available for studying linear differential equations, one particularly useful tool is provided by Laplace Transforms.

مطالعه معادلات دیفرانسیل مرتبه بالای خطی فوق موضوع بسیار جالبی بوده و یک روش خاص حل آن استفاده از تبدیل لاپلاس است.

Table : *Laplace transform table*

جدول تبدیل لاپلاس

$f(t) \quad (t \geq 0)$	$\mathcal{L}[f(t)]$	Region of Convergence
1	$\frac{1}{s}$	$\sigma > 0$
$\delta_D(t)$	1	$\lvert\sigma\rvert < \infty$
t	$\frac{1}{s^2}$	$\sigma > 0$
$t^n \quad n \in \mathbb{Z}^+$	$\frac{n!}{s^{n+1}}$	$\sigma > 0$
$e^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{s-\alpha}$	$\sigma > \Re\{\alpha\}$
$te^{\alpha t} \quad \alpha \in \mathbb{C}$	$\frac{1}{(s-\alpha)^2}$	$\sigma > \Re\{\alpha\}$
$\cos(\omega_o t)$	$\frac{s}{s^2+\omega_o^2}$	$\sigma > 0$
$\sin(\omega_o t)$	$\frac{\omega_o}{s^2+\omega_o^2}$	$\sigma > 0$
$e^{\alpha t}\sin(\omega_o t+\beta)$	$\frac{(\sin\beta)s+\omega_o^2\cos\beta-\alpha\sin\beta}{(s-\alpha)^2+\omega_o^2}$	$\sigma > \Re\{\alpha\}$
$t\sin(\omega_o t)$	$\frac{2\omega_o s}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
$t\cos(\omega_o t)$	$\frac{s^2-\omega_o^2}{(s^2+\omega_o^2)^2}$	$\sigma > 0$
u(t) - u(t-τ)	$\frac{1-e^{-s\tau}}{s}$	$\lvert\sigma\rvert < \infty$

Table : *Laplace transform properties.*

خواص تبدیل لاپلاس

$f(t)$	$\mathcal{L}[f(t)]$	Names
$\sum_{i=1}^{l} a_i f_i(t)$	$\sum_{i=1}^{l} a_i F_i(s)$	Linear combination
$\frac{dy(t)}{dt}$	$sY(s) - y(0^-)$	Derivative Law
$\frac{d^k y(t)}{dt^k}$	$s^k Y(s) - \sum_{i=1}^{k} s^{k-i} \left. \frac{d^{i-1} y(t)}{dt^{i-1}} \right\|_{t=0^-}$	High order derivative
$\int_{0^-}^{t} y(\tau) d\tau$	$\frac{1}{s} Y(s)$	Integral Law
$y(t-\tau)\, u(t-\tau)$	$e^{-s\tau} Y(s)$	Delay
$ty(t)$	$-\frac{dY(s)}{ds}$	
$t^k y(t)$	$(-1)^k \frac{d^k Y(s)}{ds^k}$	
$\int_{0^-}^{t} f_1(\tau) f_2(t-\tau) d\tau$	$F_1(s) F_2(s)$	Convolution
$\lim_{t\to\infty} y(t)$	$\lim_{s\to 0} sY(s)$	Final Value Theorem
$\lim_{t\to 0^+} y(t)$	$\lim_{s\to\infty} sY(s)$	Initial Value Theorem
$f_1(t) f_2(t)$	$\frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F_1(\zeta) F_2(s-\zeta) d\zeta$	Time domain product
$e^{at} f_1(t)$	$F_1(s-a)$	Frequency Shift

Transfer Function model مدل تابع انتقال

$$\frac{d^n y}{dt^n} + a_{n-1}\frac{d^{n-1}y}{dt^{n-1}} + \ldots\ldots + a_1\frac{dy}{dt} + a_0 y = \frac{d^m u}{dt^m} + b_{m-1}\frac{d^{m-1}u}{dt^{m-1}} + \ldots\ldots + b_1\frac{du}{dt} + b_0 u$$

Taking laplace transform $\Downarrow$ *zero initial condition*

$$s^n y(s) + a_{n-1}s^{n-1}y(s) + \ldots\ldots + a_1 s y(s) + a_0 y(s) = s^m u(s) + b_{m-1}s^{m-1}u(s) + \ldots\ldots + b_1 s u(s) + b_0 u(s)$$

$$A(s)y(s) = B(s)u(s) \quad \Rightarrow \quad G(s) = \frac{y(s)}{u(s)} = \frac{B(s)}{A(s)}$$

$$A(s) = s^n + a_{n-1}s^{n-1} + \ldots\ldots + a_1 s + a_0$$

$$B(s) = s^m + b_{m-1}s^{m-1} + \ldots\ldots + b_1 s + b_0$$

TF model

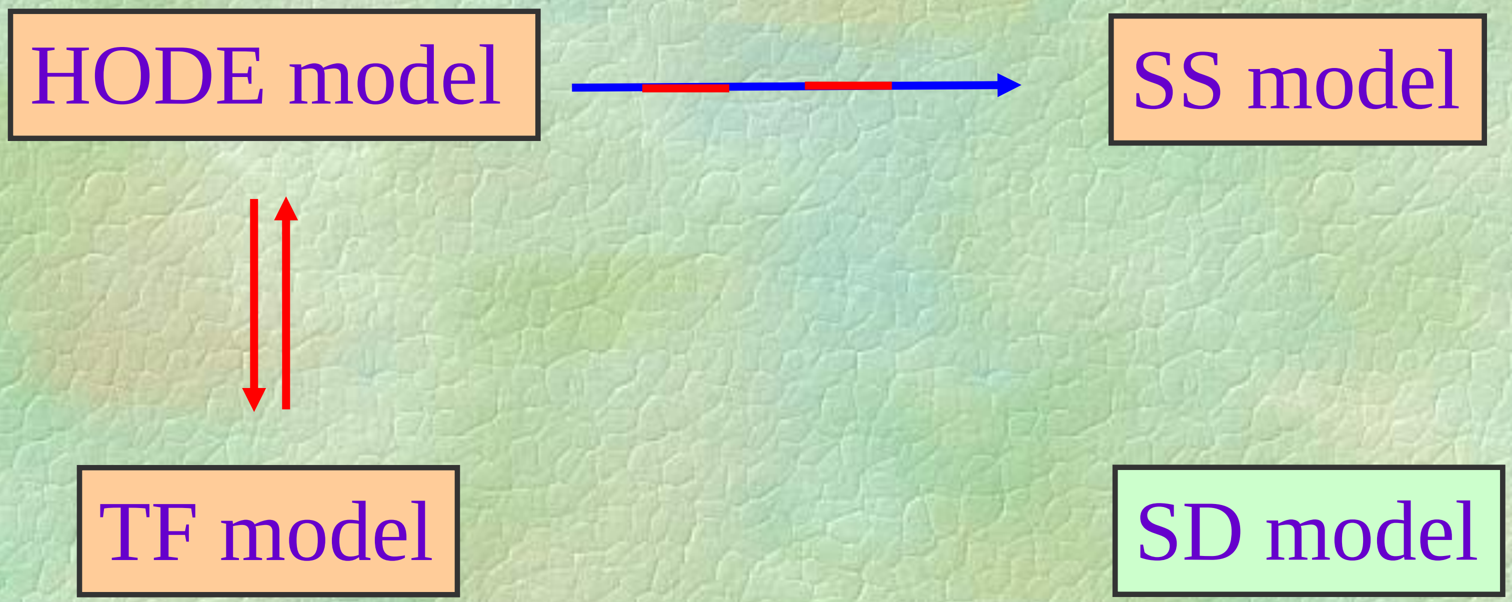
Or

Input-output model

HODE model ⟷ TF model

Different representations

نمایشهای مختلف



But how can we change SS model to TF model?

اما چگونه معادلات فضای حالت را به تابع انتقال تبدیل کنیم

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

Use Laplace transform

$$sX(s) - x(0) = \mathbf{A}X(s) + \mathbf{B}U(s)$$
$$Y(s) = \mathbf{C}X(s) + \mathbf{D}U(s)$$

Then

$$X(s) = (s\mathbf{I} - \mathbf{A})^{-1}x(0) + (s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B}U(s)$$
$$Y(s) = [\mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}]U(s) + \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}x(0)$$

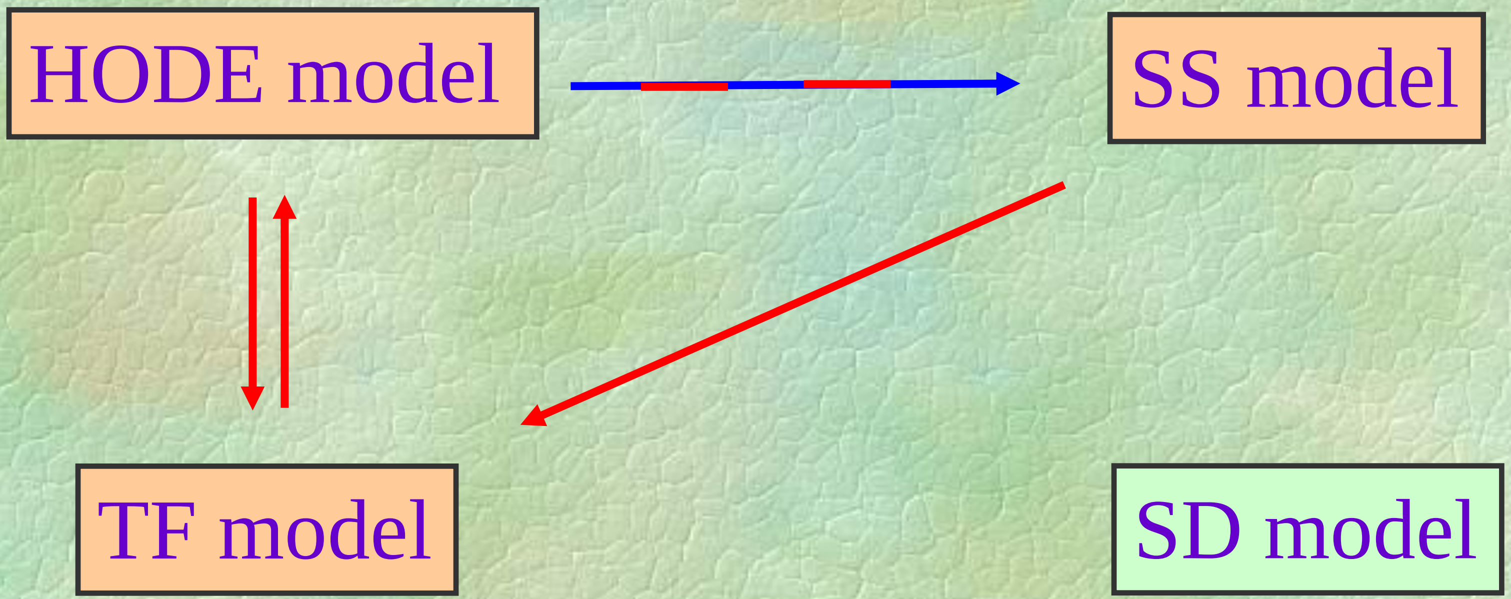
Let initial condition zero

$$Y(s) = \mathbf{G}(s)U(s)$$
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

SS model → TF model

Different representations

نمایشهای مختلف



TF model properties

خواص مدل تابع انتقال

1- It is available just for linear systems.

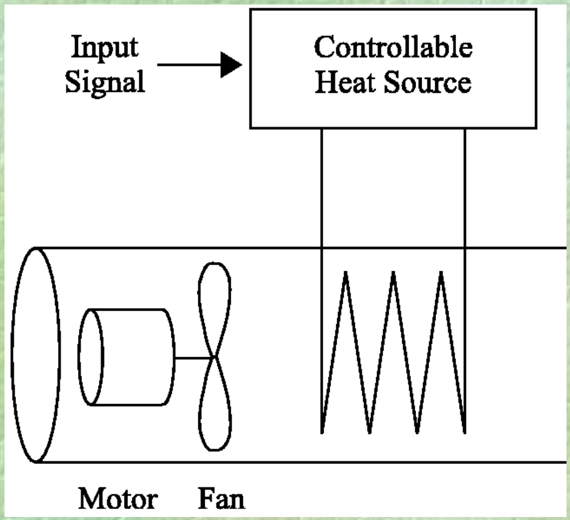
2- It is derived by zero initial condition.

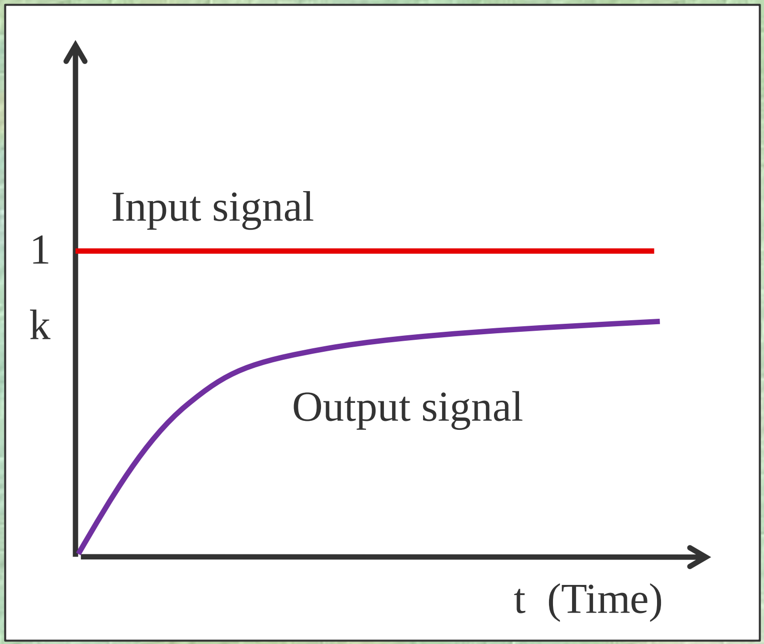
3- It just shows the relation between input and output so it may lose some information.

4- It can be used to show delay systems but SS can not.

Example 5: A thermal system

مثال۵: یک سیستم حرارتی





Input signal is an step function so:

$$u(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$$

$$u(s)=\frac{1}{s}$$

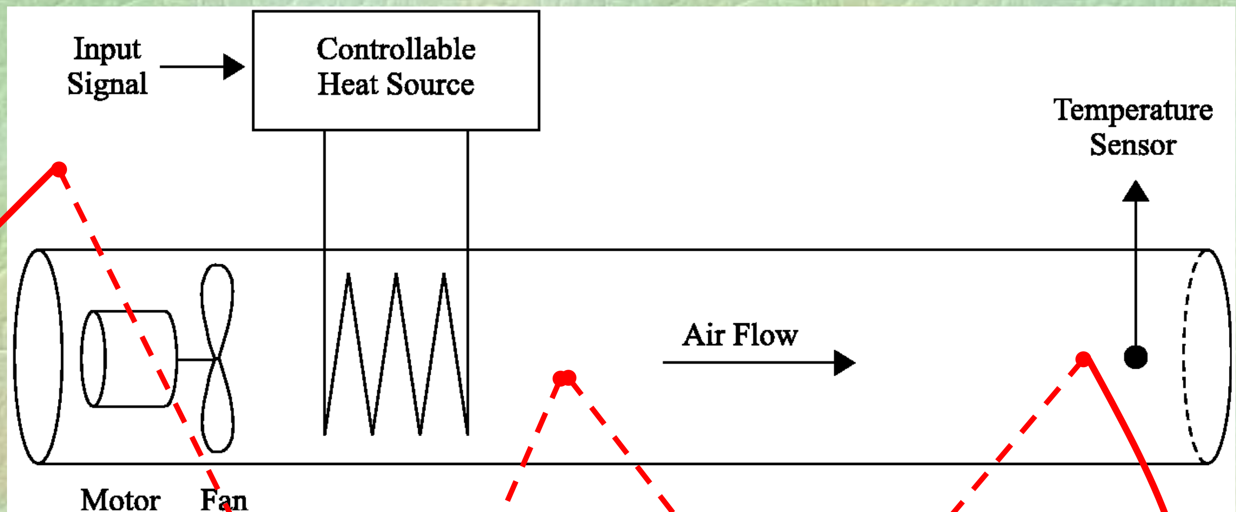
$$y(t)=\begin{cases}k-ke^{\frac{-t}{\tau}} & t\geq 0\\ 0 & t<0\end{cases}$$

$$y(s)=\frac{k}{s}-\frac{k}{s+1/\tau}$$

$$\Rightarrow \quad g(s)=\frac{y(s)}{u(s)}=\frac{k}{\tau s+1}$$

Example 6: A system with pure time delay

مثال ۶: سیستمی با تاخیر ثابت



$$g(s)=\frac{k}{\tau s+1}$$

$$h(s)=e^{-sT_d}$$

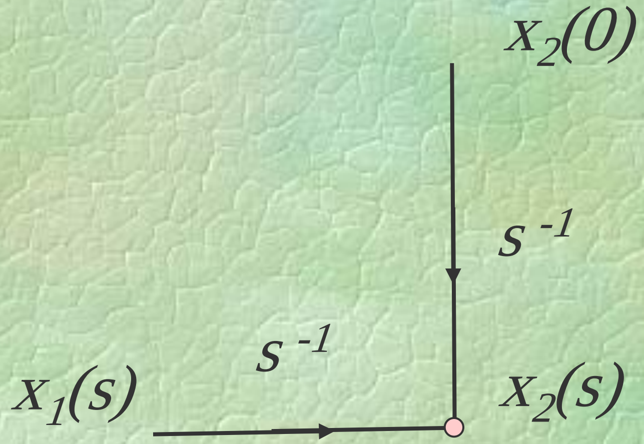
$$h(s)g(s)=\frac{ke^{-sT_d}}{\tau s+1}$$

State diagram

دیاگرام حالت

$$x_1 = \frac{dx_2}{dt} \qquad \Rightarrow \qquad x_1(s) = sx_2(s) - x_2(0)$$

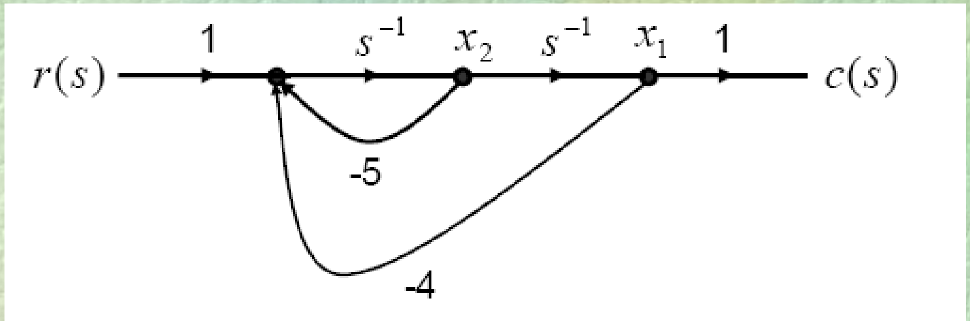
$$x_2(s) = s^{-1}\left(x_1(s) + x_2(0)\right)$$



SD model

State diagram to state space

دیاگرام حالت به معادلات حالت

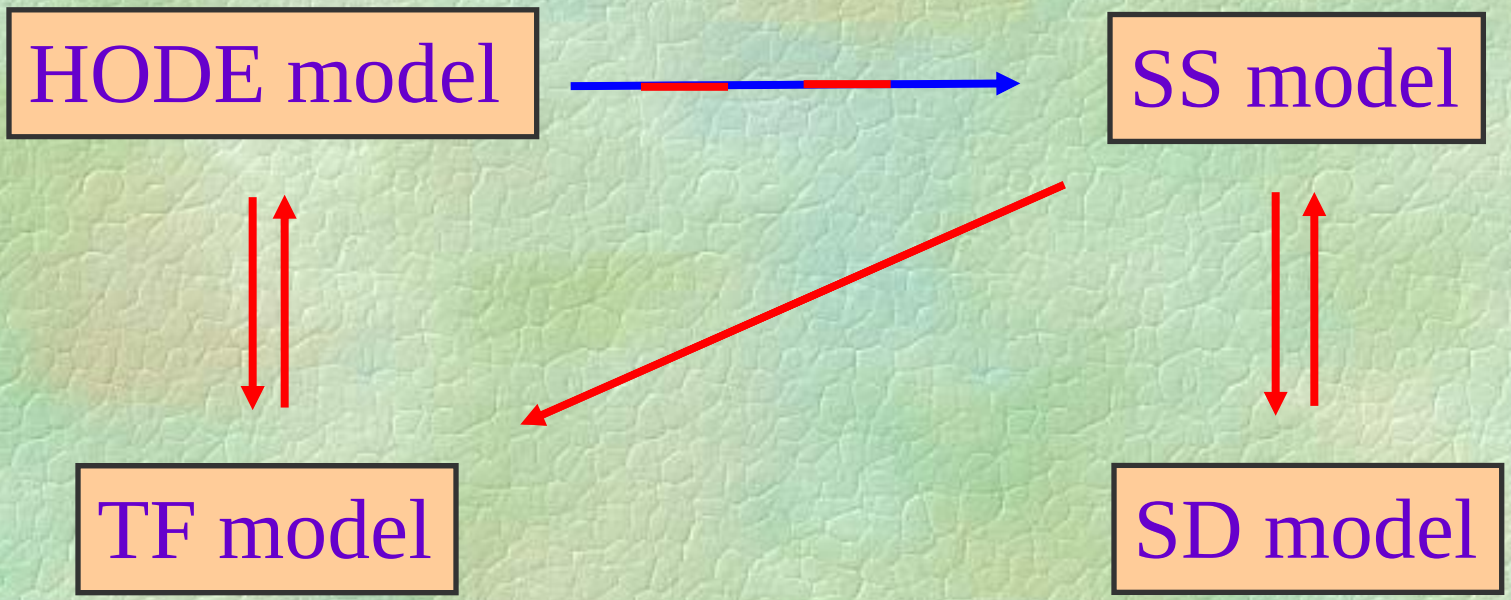


$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

SD model ⟷ SS model

Different representations

نمایشهای مختلف



Different representations

نمایشهای مختلف

HODE

$$\ddot{c}+5\dot{c}+4c=r$$

SS

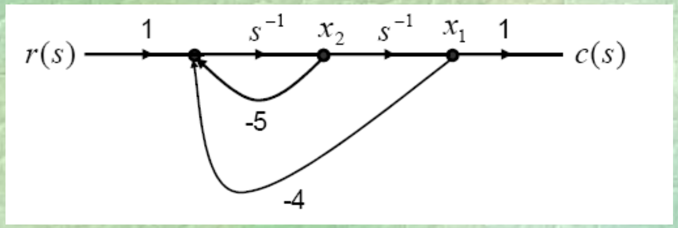
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

TF

$$G(s)=\frac{1}{s^2+5s+4}$$

Realization

SD



Mason's rule

Discussed in this lecture

Will be discussed in next lecture

Exercises

2-1 Find the SS model and TF function model for example 1.

2-2 Find the SS model and TF function model for example 2.

2-3 Find the TF function model for example 3.

2-4 Find the TF function model for example 4.
a) Suppose angular position as output
b) Suppose angular velocity as output

2-5 In example 5 find the output
a) the input is e^{-2t}
b) the input is unit step

Exercises (Continue)

2-6 In the different representation slide show the validity of red directions

2-7 Change the equations derived in example 2 to one order 3 HODE.