LINEAR CONTROL SYSTEMS

Ali Karimpour
Associate Professor
Ferdowsi University of Mashhad

Lecture 4

Converting of different representations of control systems

Topics to be covered include:

- Mason’s flow graph loop rule (Converting SD to TF)
- Realization (Converting TF to SS model)
- Converting high order differential equation to state space

Different representations

نمایشهای مختلف

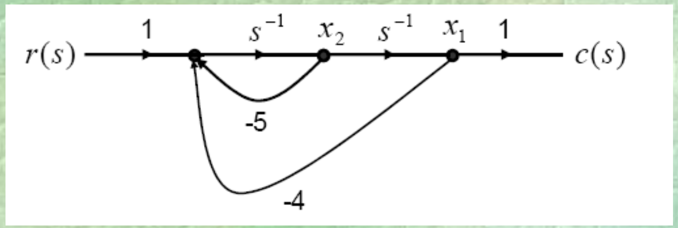
HODE

$$\ddot{c}+5\dot{c}+4c=r$$

SS

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

SD



Realization

TF

$$G(s)=\frac{c(s)}{r(s)}=\frac{1}{s^2+5s+4}$$

Mason's rule

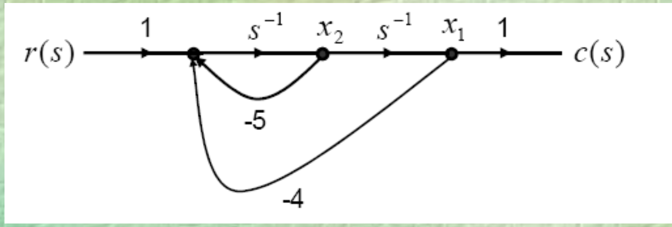
Discussed in the last lecture

Will be discussed in this lecture

Mason's flow graph loop rule

فرمول بهره میسون

SD



Mason's rule

TF

$$G(s)=\frac{c(s)}{r(s)}=\frac{1}{s^2+5s+4}$$

Mason's flow graph lop rule

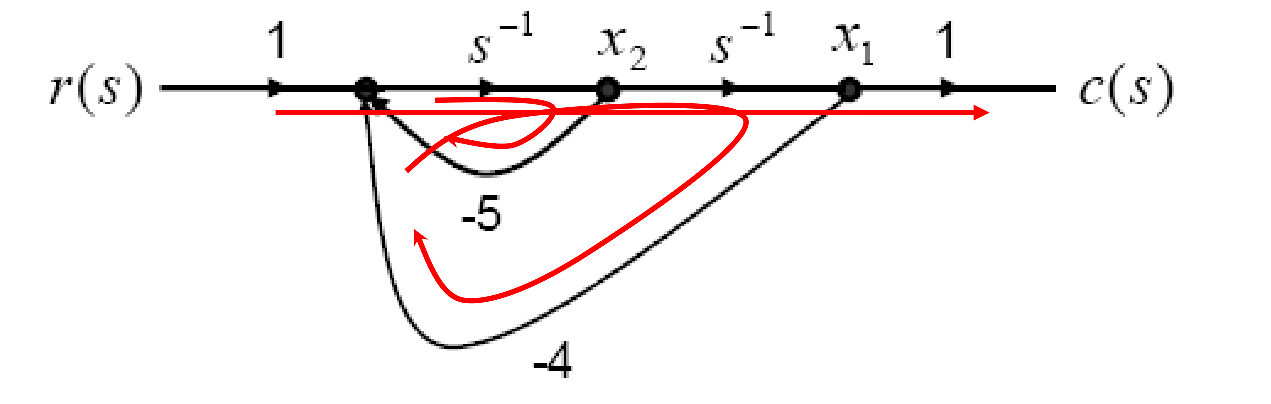
قانون گین میسون

$$\frac{y(s)}{u(s)} = \frac{\sum_{i=1}^{m} M_i \Delta_i}{\Delta}$$

m	is the number of path from input to output
M_i	gain of i^{th} path
Δ	loop gain of system
Δ_i	loop gain of system after deleting i^{th} path

$$\Delta = 1 - \sum_{m1} P_{m1} + \sum_{m2} P_{m2} - \sum_{m3} P_{m3} + - \ldots..$$

Example 1 مثال ١



$$\frac{c(s)}{r(s)} = \frac{\sum_{i=1}^{m} M_i \Delta_i}{\Delta}$$

$m = 1$ $M_1 = s^{-2}$ $\Delta_1 = 1$

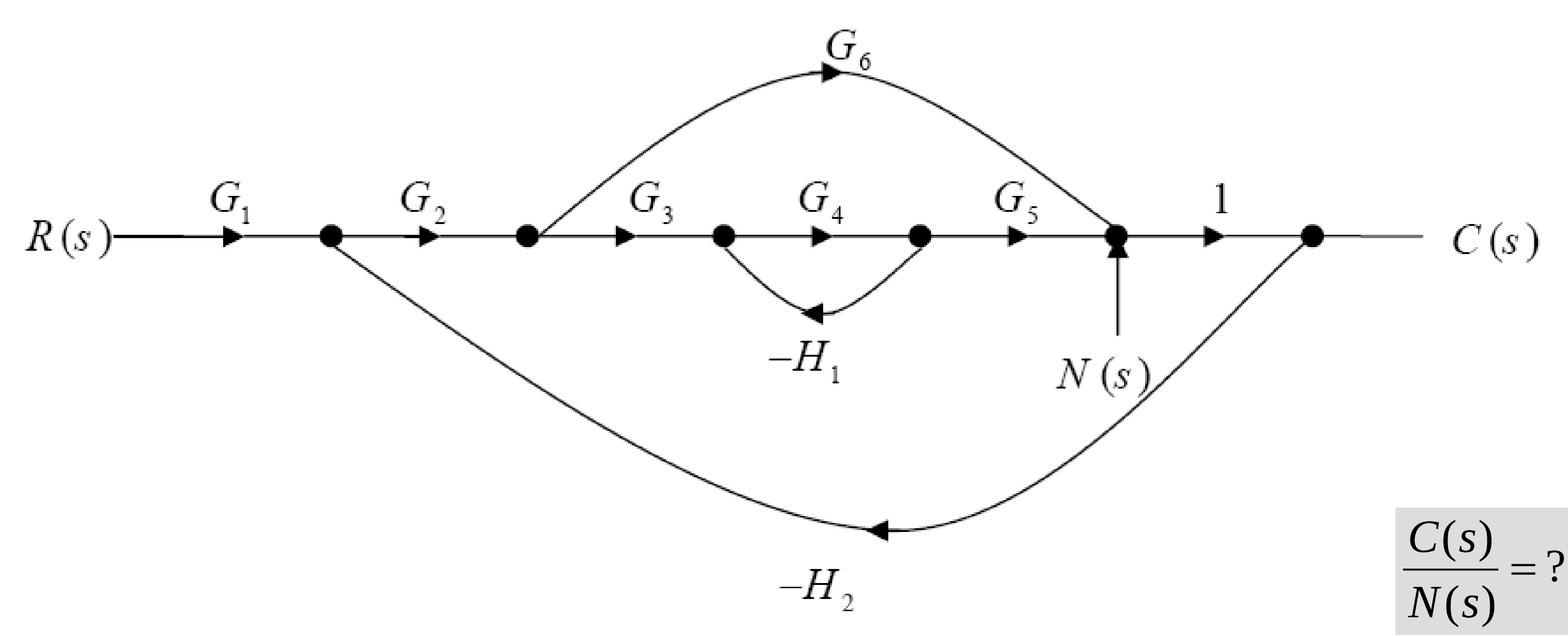
$$\Delta = 1 - \sum_{m1} P_{m1} + \sum_{m2} P_{m2} - \sum_{m3} P_{m3} + - \ldots\ldots =$$

$$1 - (-5s^{-1} - 4s^{-2}) + 0 - 0 + \ldots\ldots\ldots = 1 + 5s^{-1} + 4s^{-2}$$

$$\frac{c(s)}{r(s)} = \frac{s^{-2}.1}{1 + 5s^{-1} + 4s^{-2}} = \frac{1}{s^2 + 5s + 4}$$

Note

Example 2 مثال ۲



$$\frac{C(s)}{N(s)} = ?$$

$m = 1$

$M_1 = 1 \quad \Delta_1 = 1 + G_4 H_1$

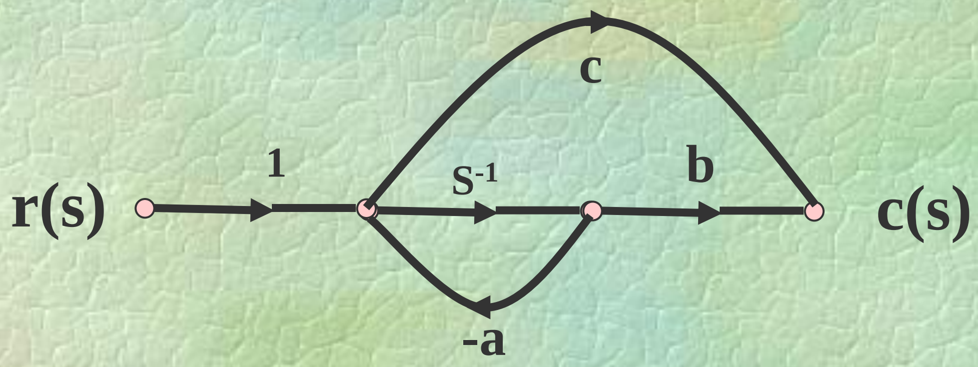
$$\Delta = 1 - (-H_1 G_4 - G_2 G_6 H_2 - G_2 G_3 G_4 G_5 H_2) + \big((-G_4 H_1)(-G_2 G_6 H_2)\big)$$
$$= 1 + H_1 G_4 + G_2 G_6 H_2 + G_2 G_3 G_4 G_5 H_2 + G_4 H_1 G_2 G_6 H_2$$

$$\frac{C(s)}{N(s)} = \frac{1.(1 + H_1 G_4)}{1 + H_1 G_4 + G_2 G_6 H_2 + G_2 G_3 G_4 G_5 H_2 + G_4 H_1 G_2 G_6 H_2}$$

Example 3

مثال ۳

Find the TF model for the following state diagram.



$$\frac{c(s)}{r(s)} = \frac{c.1 + bs^{-1}.1}{1+as^{-1}} = \frac{cs+b}{s+a}$$

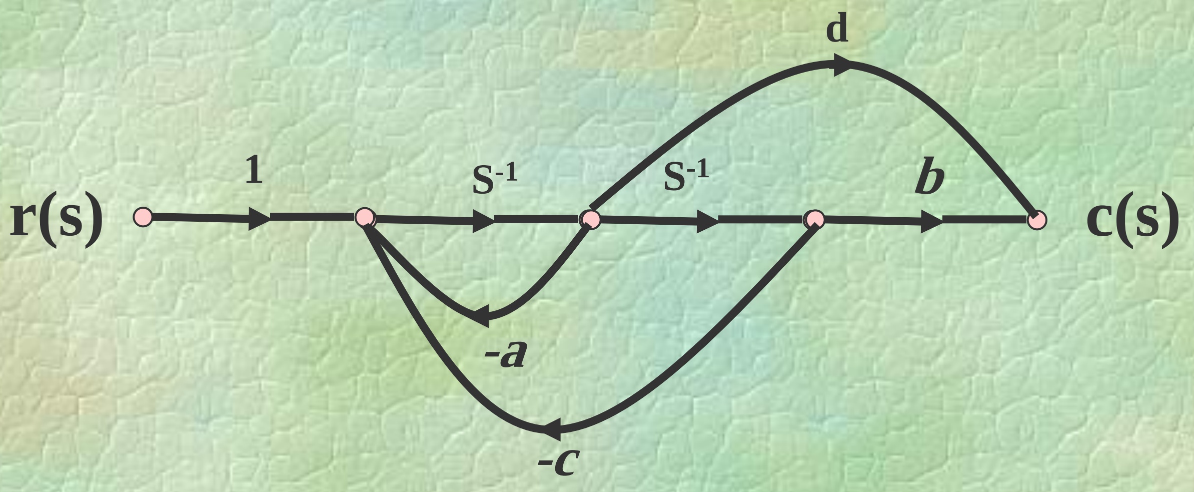
This is a base form for order 1 transfer function.

این یک حالت پایه برای تابع انتقال درجه ۱ است

Example 3

مثال ۳

Find the TF model for following state diagram.



$$\frac{c(s)}{r(s)}=\frac{ds^{-1}+bs^{-2}}{1+as^{-1}+cs^{-2}}=\frac{ds+b}{s^2+as+c}$$

This is a base form for order 2 transfer function.

این یک حالت پایه برای تابع انتقال درجه ۲ است

Realization

پیاده سازی

TF

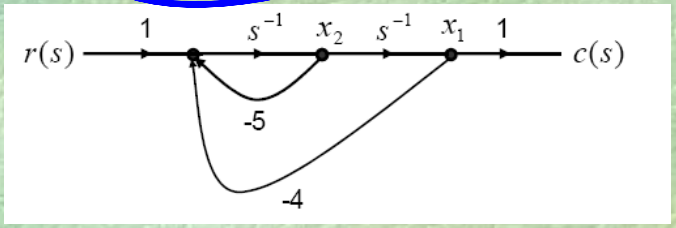
$$G(s)=\frac{1}{s^2+5s+4}$$

Realization

SS

$$\dot{x}_1=x_2$$
$$\dot{x}_2=-4x_1-5x_2+r$$
$$c=x_1$$

SD



Realization

پیاده سازی

Some Realization methods

چند روش پیاده سازی

1- Direct Realization.

۱– پیاده سازی مستقیم

2- Series Realization.

۲– پیاده سازی سری

3- Parallel Realization.

۳– پیاده سازی موازی

Direct Realization

پیاده سازی مستقیم

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}=\frac{s^{-1}+7s^{-2}+12s^{-3}}{1+4s^{-1}+5s^{-2}+2s^{-3}}\frac{X}{X}$$

$$X(1+4s^{-1}+5s^{-2}+2s^{-3})=r(s) \quad \rightarrow \quad X=r(s)-4s^{-1}X-5s^{-2}X-2s^{-3}X$$

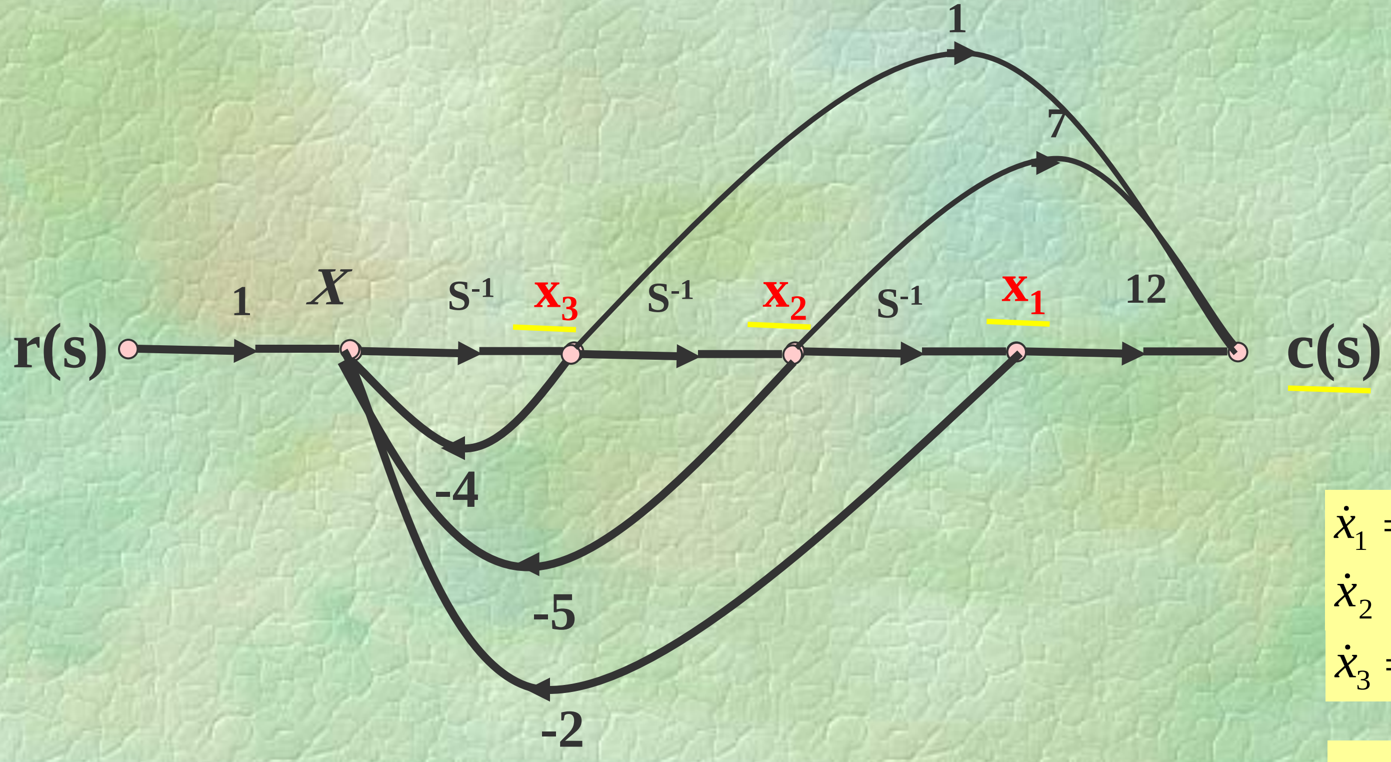
$$s^{-1}X+7s^{-2}X+12s^{-3}X=c(s)$$

پیاده سازی مستقیم

Direct Realization

$$X = r(s) - 4s^{-1}X - 5s^{-2}X - 2s^{-3}X$$

$$c(s) = 12s^{-3}X + 7s^{-2}X + s^{-1}X$$



$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = x_3$$

$$\dot{x}_3 = -2x_1 - 5x_2 - 4x_3 + r$$

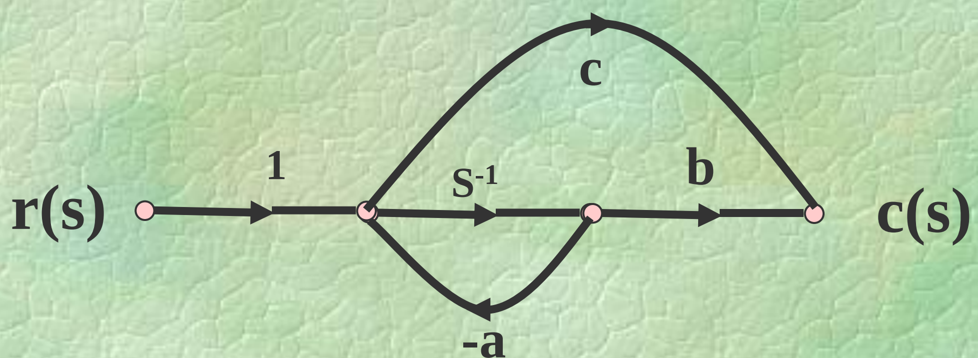
$$c = 12x_1 + 7x_2 + x_3$$

پیاده سازی سری

Series Realization

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}=\frac{s+3}{s+1}\cdot\frac{s+4}{s+1}\cdot\frac{1}{s+2}$$

$$\frac{cs+b}{s+a}$$

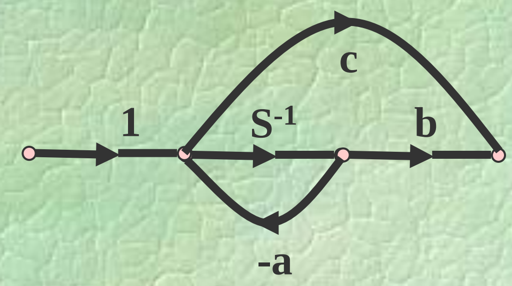


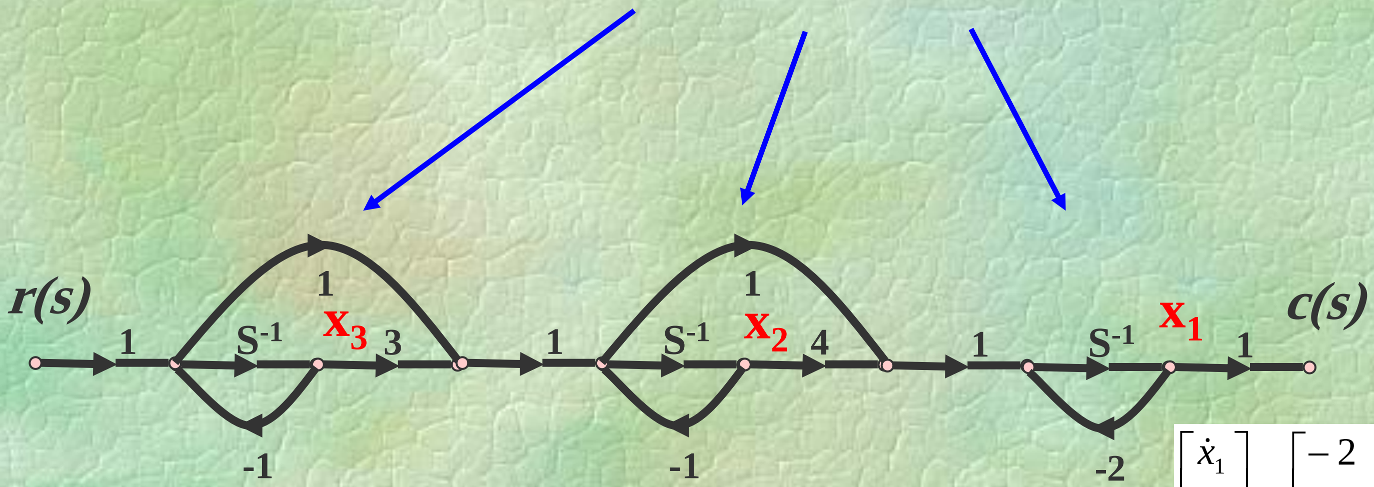
پیاده سازی سری

Series Realization

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}=\frac{s+3}{s+1}\cdot\frac{s+4}{s+1}\cdot\frac{1}{s+2}$$

$$\frac{cs+b}{s+a}$$





$$\dot{x}_1=-2x_1+4x_2-x_2+3x_3-x_3+r$$

$$\dot{x}_2=-x_2+3x_3-x_3+r$$

$$\dot{x}_3=-x_3+r$$

$$c=x_1$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\\ \dot{x}_3\end{bmatrix}=\begin{bmatrix}-2 & 3 & 2\\ 0 & -1 & 2\\ 0 & 0 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}+\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}r$$

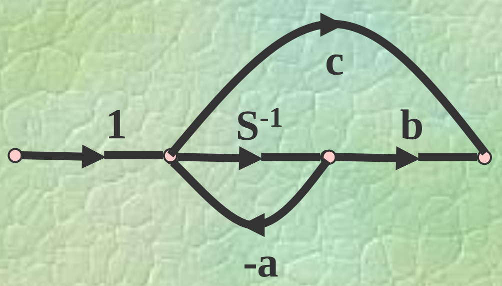
$$c=\begin{bmatrix}1 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}$$

Parallel Realization

پیاده سازی موازی

$$G(s)=\frac{c(s)}{r(s)}=\frac{2s^2+13s+17}{s^3+6s^2+11s+6}=\frac{2s^2+13s+17}{(s+1)(s+2)(s+3)}=\frac{3}{s+1}+\frac{1}{s+2}+\frac{-2}{s+3}$$

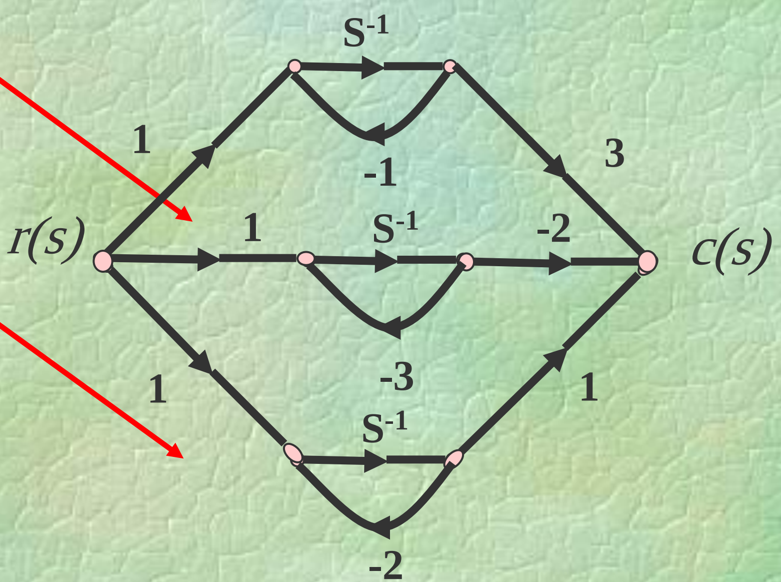
$$\frac{cs+b}{s+a}$$



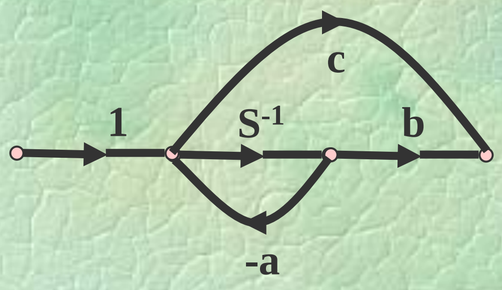
پیاده سازی موازی

Parallel Realization

$$G(s)=\frac{3}{s+1}+\frac{1}{s+2}+\frac{-2}{s+3}$$



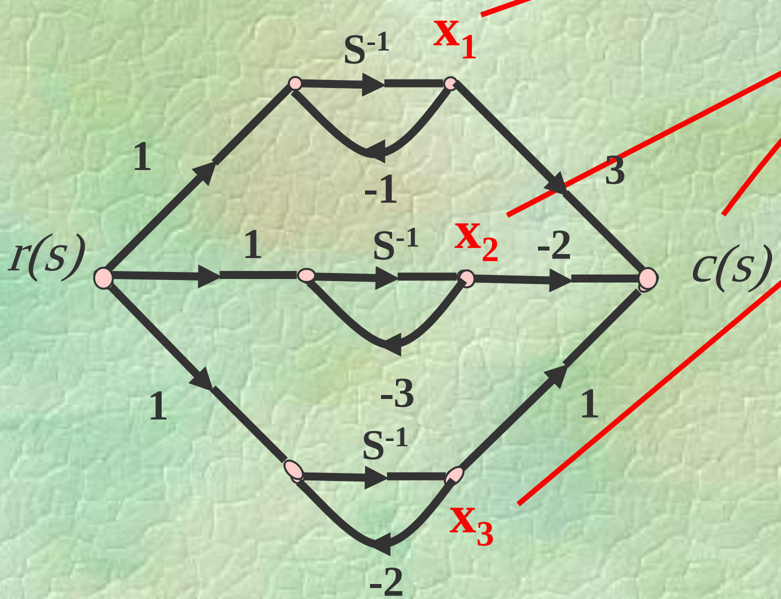
$$\frac{cs+b}{s+a}$$



پیاده سازی موازی

Parallel Realization

$$G(s)=\frac{3}{s+1}+\frac{1}{s+2}+\frac{-2}{s+3}$$



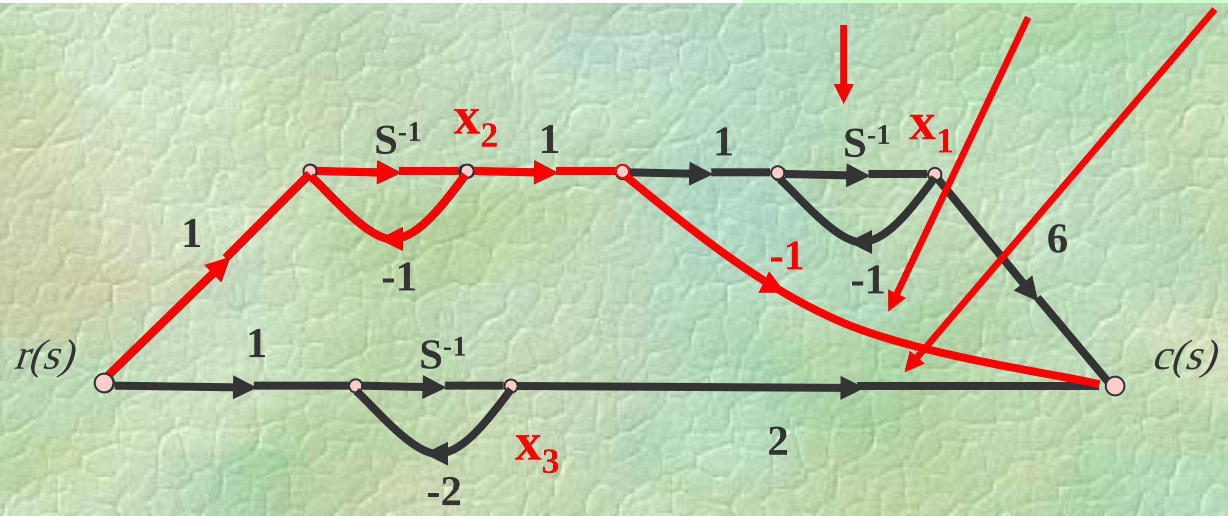
$$\dot{x}_1=-x_1+r$$
$$\dot{x}_2=-3x_2+r$$
$$\dot{x}_3=-2x_3+r$$
$$c=3x_1-2x_2+x_3$$

$$\dot{x}=\begin{bmatrix}-1&0&0\\0&-3&0\\0&0&-2\end{bmatrix}x+\begin{bmatrix}1\\1\\1\end{bmatrix}u$$
$$c=[3\quad -2\quad 1]x$$

Parallel Realization

پیاده سازی موازی

$$G(s)=\frac{c(s)}{r(s)}=\frac{s^2+7s+12}{s^3+4s^2+5s+2}=\frac{s^2+7s+12}{(s+1)^2(s+2)}=\frac{6}{(s+1)^2}+\frac{-1}{s+1}+\frac{2}{s+2}$$



$$\dot{x}_1=-x_1+x_2$$
$$\dot{x}_2=-x_2+r$$
$$\dot{x}_3=-2x_3+r$$
$$c=6x_1-1x_2+2x_3$$

$$\dot{x}=\begin{bmatrix}-1 & 1 & 0\\ 0 & -1 & 0\\ 0 & 0 & -2\end{bmatrix}x+\begin{bmatrix}0\\1\\1\end{bmatrix}r$$
$$c=[6\quad -1\quad 2]x$$

Converting high order differential equation to state space

HODE

$$\ddot{x}_1 + 5\dot{x}_1 + 4x_1 = r$$

SS

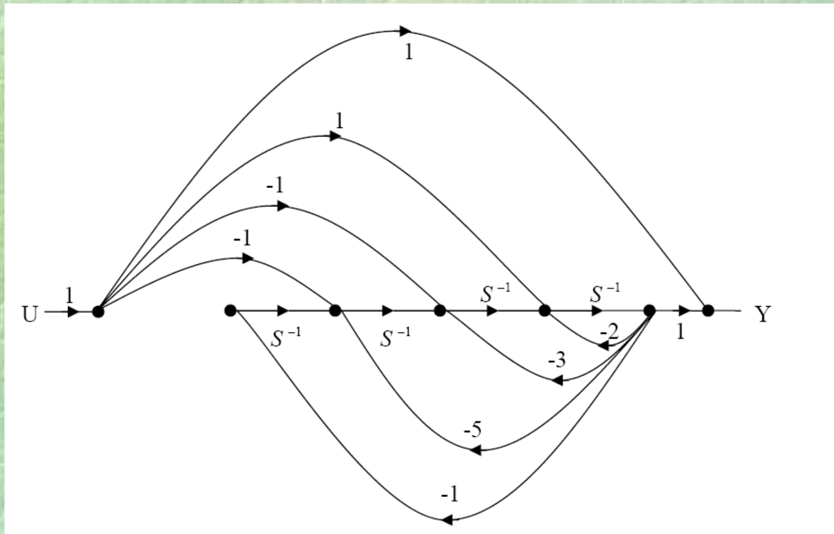
$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -4x_1 - 5x_2 + r$$
$$c = x_1$$

TF

$$G(s) = \frac{1}{s^2 + 5s + 4}$$

Exercises

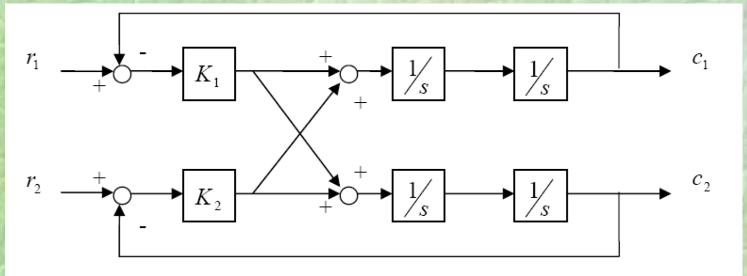
4-1 Find *Y(s)/U(s)* for following system



$$Ans = \frac{S^4 + 3S^3 + 2S^2 + 4S + 1}{S^4 + 2S^3 + 3S^2 + 5S + 1}$$

Exercises (Continue)

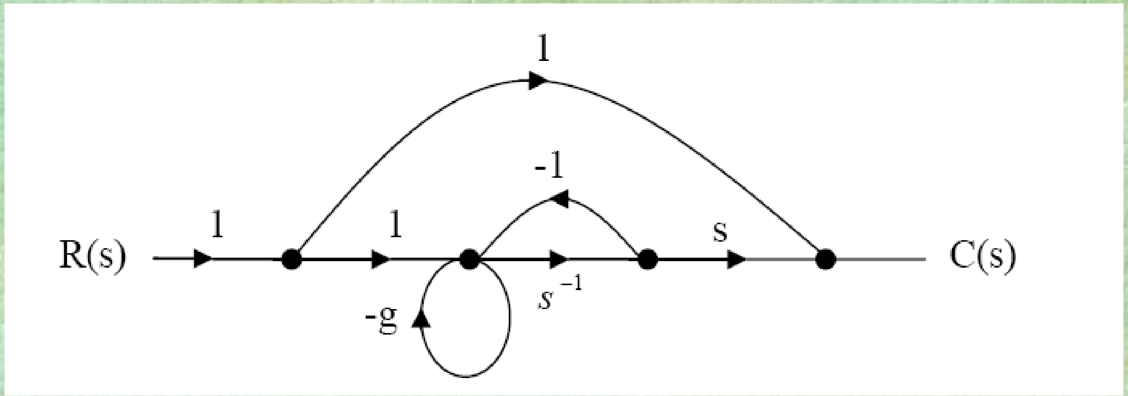
4-2 Find $c_1(s)/r_2(s)$ for following system



Answer

$$\frac{c_1}{r_2} = \frac{k_2}{s^2 + k_1 + k_2}$$

4-3 Find g such that C(s)/R(s) for following system is (2s+2)/s+2

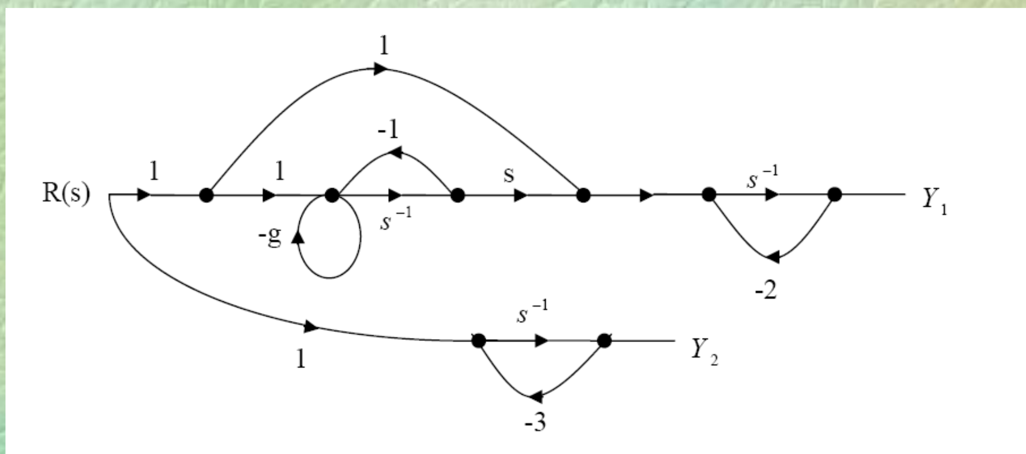


Answer

$$g = \frac{1}{s}$$

Exercises (Continue)

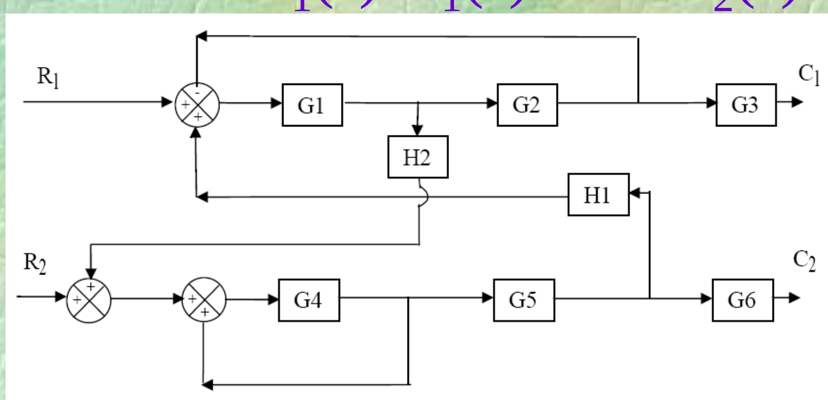
4-4 Find $Y_1(s)/R(s)$ for following system



Answer

$$\frac{(g+2)s^2+(3g+7)s+3}{(g+1)s^3+(5g+6)s^2+(6g+11)s+6}$$

4-5 Find $C_1(s)/R_1(s)$ and $C_2(s)/R_2(s)$ for following system



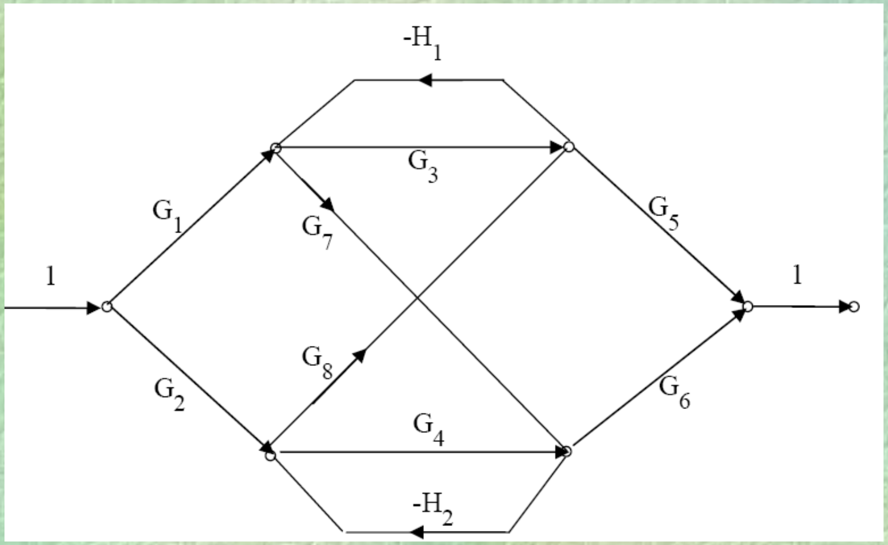
Answer

$$\frac{C_2(S)}{R_2(S)}=\frac{P_2\Delta_2}{\Delta}=\frac{G_4G_5G_6(1+G_1G_2)}{1-G_4+G_1G_2-G_1H_2G_4G_5H_1-G_1G_2G_4}$$

$$\frac{C_1(S)}{R_1(S)}=\frac{P_1\Delta_1}{\Delta}=\frac{G_1G_2G_3(1-G_4)}{1-G_4+G_1G_2-G_1H_2G_4G_5H_1-G_1G_2G_4}$$

Exercises (Continue)

4-6 Find C/R for following system



Answer

$$\frac{C}{R}=\frac{G_1G_3G_5(1+G_4H_2)+G_2G_4G_6(1+G_3H_1)+G_1G_7G_6+G_2G_8G_5-G_1G_7H_2G_8G_5-G_2G_8H_1G_7G_6}{1+G_4H_2+G_3H_1-G_7H_2G_8H_1+G_4H_2G_3H_1}$$

Exercises (Continue)

4-7 Find the SS model for following system.

$$y'' + 5y' + 6y = u' + u$$

It was in some exams.

4-8 Find the TF model for following system without any inverse manipulation.

$$\dot{x} = \begin{bmatrix} -1 & 3 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x$$

4-9 Find the SS model for following system.

$$G(s) = \frac{S^4 + 3S^3 + 2S^2 + 4S + 1}{S^4 + 2S^3 + 3S^2 + 5S + 1}$$

Exercises (Continue)

4-10 Find direct realization, series realization and parallel realization for following system.

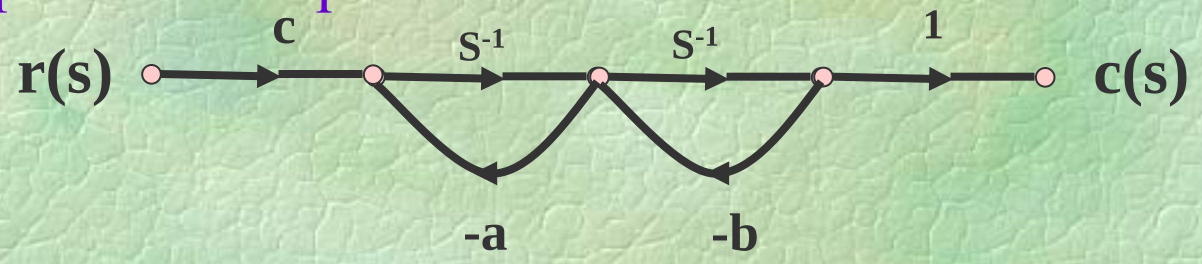
$$G(s)=\frac{s+1}{(s^2+2s+2)(s+3)}$$

4-11 a) Find the transfer function of following system by Masson formula.

b) Find the state space model of system.

c) Find the transfer function of system from the SS model in part b

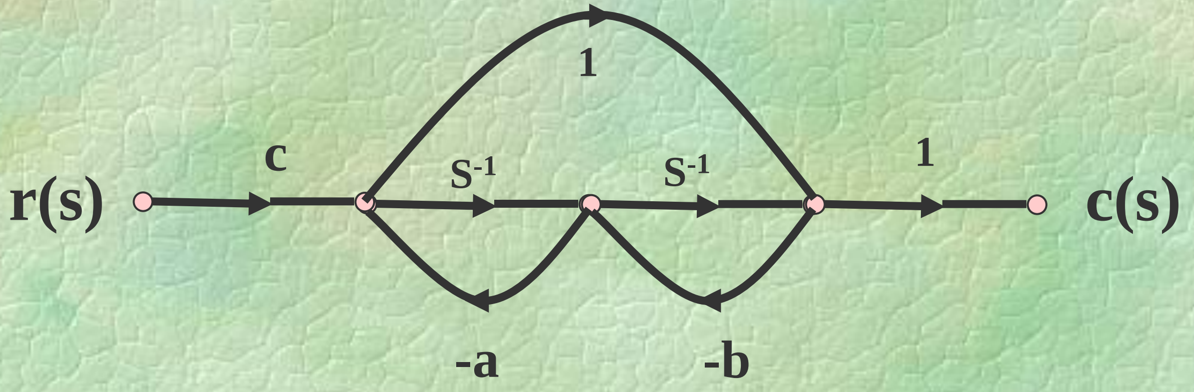
d) Compare part “a” and part “c”



It appears in many exams.*****

Exercises (Continue)

4-12 a) Find the transfer function of following system by Masson formula.
b) Find the state space model of system.
c) Find the transfer function of system from the SS model in part b
d) Compare part "a" and part "c"



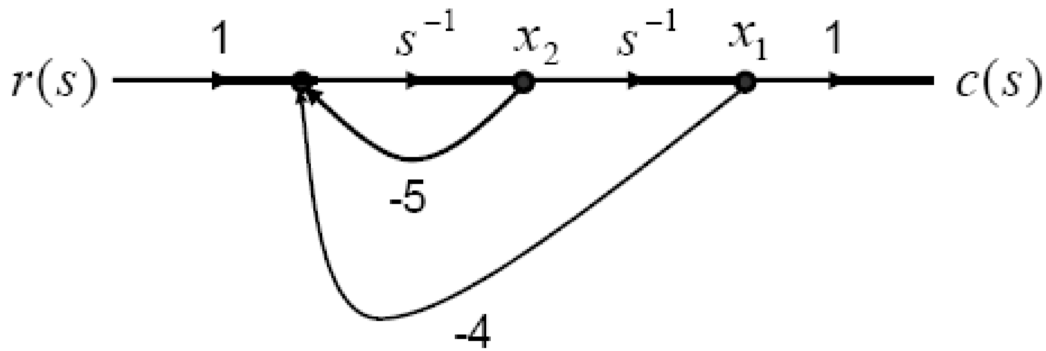
Regional Electrical engineering Olympiad spring 2008.*****

Appendix: Example 1 a) Draw the state diagram for the following differential equation. b) Suppose c(t) as output and r(t) as input and find transfer function.

$$\frac{d^2c(t)}{dt^2}+5\frac{dc(t)}{dt}+4c(t)=r(t)$$

Solution: a) Consider $x_1(t)=c(t),\ x_2(t)=c'(t)$ then we have:

$x_1'(t)=c'(t)=x_2(t)$ and $x_2'(t)=c''(t)=r(t)-4c(t)-5c'(t)=r(t)-4x_1(t)-5x_2(t)$ so:



b) By general gain formula, transfer function is:

$$\frac{c(s)}{r(s)}=\frac{M_1\Delta_1}{\Delta}=\frac{\left(s^{-2}\right)\times 1}{1-(-5s^{-1}-4s^{-2})}=\frac{1}{s^2+55+4}$$

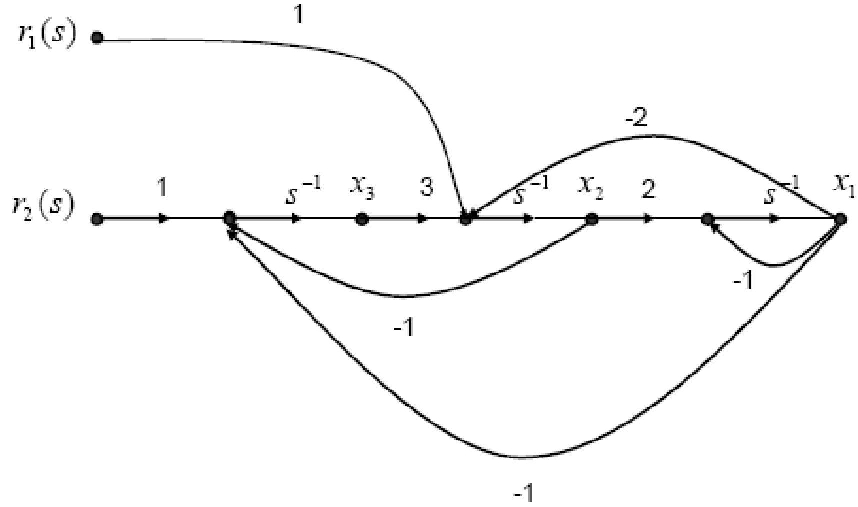
Appendix: Example 2: Express the following set of differential equations in the form of $\dot{X}(t) = AX(t) + Br(t)$ and draw corresponding state diagram.

$$\dot{x}_1 = -x_1(t) + 2x_2(t)$$
$$\dot{x}_2 = -2x_1(t) + 3x_3(t) + r_1(t)$$
$$\dot{x}_3 = -x_1(t) - x_2(t) + r_2(t)$$

Solution: Clearly

$$\dot{X} = \begin{bmatrix} -1 & 2 & 0 \\ -2 & 0 & 3 \\ -1 & -1 & 0 \end{bmatrix} X + \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} r_1(t) \\ r_2(t) \end{bmatrix}$$

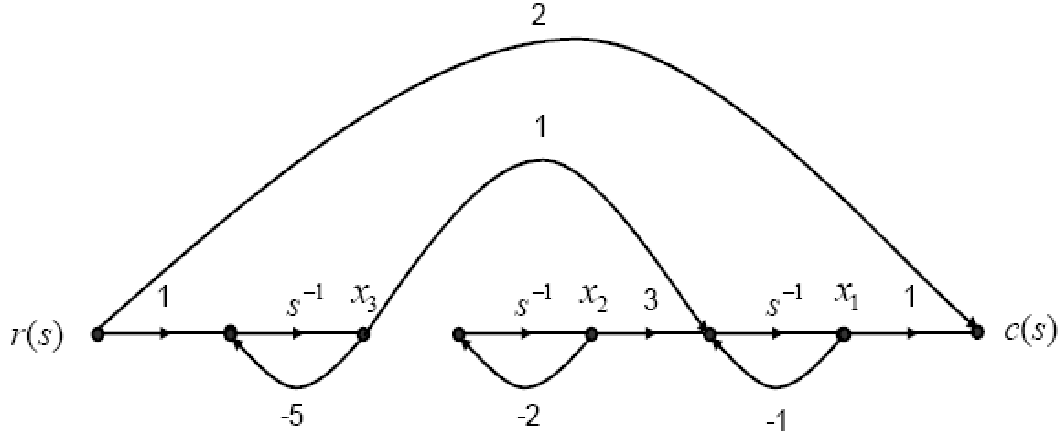
State diagram is:



Appendix: Example 3: Determine the transfer function of following system without using any inverse manipulation.

$$\dot{X}(t)=\begin{bmatrix}-1 & 3 & 1\\ 0 & -2 & 0\\ 0 & 0 & -5\end{bmatrix}X(t)+\begin{bmatrix}0\\0\\1\end{bmatrix}r(t) \qquad c(t)=[1 \quad 0 \quad 0]X(t)+2r(t)$$

Solution: Transfer function without inverse manipulation is possible by using state diagram, state diagram of system is:



By using general gain formula, transfer function is:

$$\frac{c(s)}{r(s)}=\frac{M_1\Delta_1+M_2\Delta_2}{\Delta}=\frac{2\times\left(1-(-5s^{-1}-2s^{-1}-s^{-1})+(10s^{-2}+5s^{-2}+2s^{-2})-(-10s^{-3})\right)+s^{-2}\times\left(1-(-2s^{-1})\right)}{1-(-5s^{-1}-2s^{-1}-s^{-1})+(10s^{-2}+5s^{-2}+2s^{-2})-(-10s^{-3})}$$

$$=\frac{2s^3+16s^2+35s+22}{s^3+8s^2+17s+10}$$

Appendix: Example 4: Determine the modal form of this transfer function. One particular useful canonical form is called the Modal Form.

It is a diagonal representation of the state-space model. Assume for now that the transfer function has distinct real poles p_i (but this easily extends to the case with complex poles.)

$$\begin{aligned} G(s) &= \frac{N(s)}{D(s)} = \frac{N(s)}{(s-p_1)(s-p_2)\cdots(s-p_n)} \\ &= \frac{r_1}{s-p_1} + \frac{r_2}{s-p_2} + \cdots + \frac{r_n}{s-p_n} \end{aligned}$$

Now define a collection of first order systems, each with state x_i

Appendix: Example 4: Determine the modal form of this transfer function. One particular useful canonical form is called the Modal Form.

$$
\begin{aligned}
\frac{X_1}{U(s)} &= \frac{r_1}{s - p_1} \Rightarrow \dot{x}_1 = p_1 x_1 + r_1 u \\
\frac{X_2}{U(s)} &= \frac{r_2}{s - p_2} \Rightarrow \dot{x}_2 = p_2 x_2 + r_2 u \\
&\vdots \\
\frac{X_n}{U(s)} &= \frac{r_n}{s - p_n} \Rightarrow \dot{x}_n = p_n x_n + r_n u
\end{aligned}
$$

Which can be written as

$$
\begin{aligned}
\dot{\mathbf{x}}(t) &= A\mathbf{x}(t) + Bu(t) \\
y(t) &= C\mathbf{x}(t) + Du(t)
\end{aligned}
$$

Appendix: Example 4: Determine the modal form of this transfer function. One particular useful canonical form is called the Modal Form.

With :

$$A = \begin{bmatrix} p_1 & & \\ & \ddots & \\ & & p_n \end{bmatrix} \quad B = \begin{bmatrix} r_1 \\ \vdots \\ r_n \end{bmatrix} \quad C = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix}^T$$

Good representation to use for numerical robustness reasons.
Avoids some of the large coefficients in the other 4 canonical forms.