LINEAR CONTROL SYSTEMS

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Lecture 5

Topics to be covered include:

- Nonlinear systems
- Linearization of nonlinear systems
- Transfer function representation
  - Property
  - Poles and zeros and their physical meaning
  - SISO and MIMO
  - Open loop and closed loop system
  - Effect of feedback
  - Characteristic equation for SISO system

Nonlinear systems. سیستمهای غیر خطی

Although almost every real system includes nonlinear features, many systems can be reasonably described, at least within certain operating ranges, by linear models.

گرچه تقریبا تمام سیستمهای واقعی دارای رفتار غیر خطی هستند، بسیاری از سیستمها را می توان حداقل در یک رنج کاری خاص خطی نمود.

Nonlinear systems. سیستمهای غیر خطی

$$\dot{x}(t) = f(x(t), u(t))$$
$$y(t) = g(x(t), u(t))$$

Say that $\{x_Q(t), u_Q(t), y_Q(t)\}$ is a given set of trajectories that satisfy the above equations, so we have

$$\dot{x}_Q(t) = f(x_Q(t), u_Q(t)); \qquad x_Q(t_o) \text{ given}$$
$$y_Q(t) = g(x_Q(t), u_Q(t))$$

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

Linearized system سیستم خطی شده

$$\dot{x}(t) \approx f(x_Q, u_Q) + \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) \approx g(x_Q, u_Q) + \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$\dot{x}(t) - f(x_Q, u_Q) \approx \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

$$y(t) - g(x_Q, u_Q) \approx \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} (x(t) - x_Q) + \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}} (u(t) - u_Q)$$

Linearization procedure

$$\dot{\delta x} = A\,\delta x + B\,\delta u$$

$$\delta y = C\,\delta x + D\,\delta u$$

$$A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

$$C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q \\ u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q \\ u=u_Q}}$$

Example 1

Consider a continuous time system with true model given by

$$\frac{dx(t)}{dt} = f(x(t), u(t)) = -\sqrt{x(t)} + \frac{(u(t))^2}{3}$$

Assume that the input $u(t)$ fluctuates around $u = 2$. Find an operating point with $u_Q = 2$ and a linearized model around it.

$$u_Q = 2 \quad \Rightarrow \quad 0 = -\sqrt{x_Q} + \frac{2^2}{3} \quad \Rightarrow \quad x_Q = \frac{16}{9}$$

Operating point: $u_Q = 2 \quad x_Q = \frac{16}{9}$

Example 1 (Continue)

$$\frac{dx(t)}{dt} = f(x(t),u(t)) = -\sqrt{x(t)} + \frac{(u(t))^2}{3}$$

Operating point:

$$u_Q = 2 \qquad x_Q = \frac{16}{9}$$

Linearization procedure

$$\dot{\delta} x = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

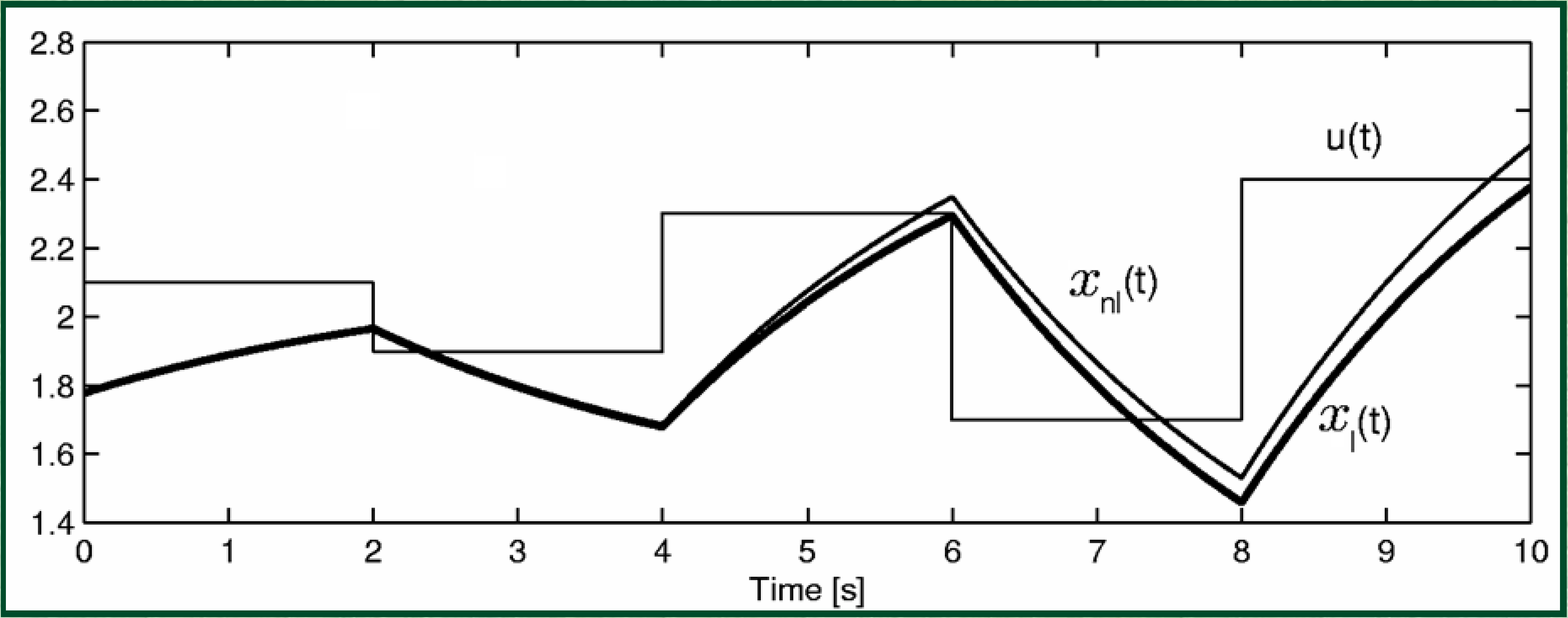
$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$A = \frac{\partial f}{\partial x}\Big|_{x=x_Q,u=u_Q} = -\frac{1}{2\sqrt{x_Q}} = -\frac{3}{8}$$

$$B = \frac{\partial f}{\partial u}\Big|_{x=x_Q,u=u_Q} = \frac{2}{3}u_Q = \frac{4}{3}$$

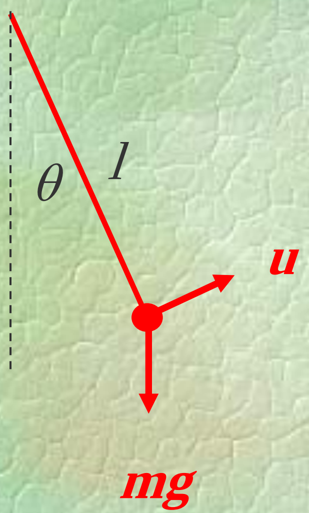
$$\dot{\delta x} = -\frac{3}{8}\delta x + \frac{4}{3}\delta u$$

Simulation شبیه سازی



Example 2 (Pendulum)

Find the linear model around equilibrium point.



$$J\frac{d^2\theta}{dt^2} = ul - mgl\sin\theta, \quad J = ml^2$$

$$\frac{d^2\theta}{dt^2} = \frac{u}{ml} - \frac{g}{l}\sin\theta \qquad x_1 = \theta, x_2 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{u}{ml} - \frac{g}{l}\sin x_1$$

$$x_{1Q} = x_{2Q} = u_Q = 0$$

is operating point

Linearization procedure

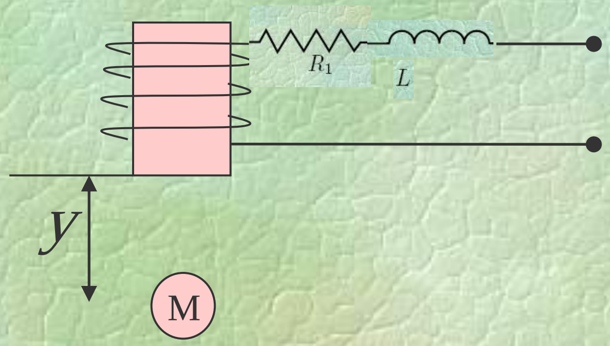
$$\dot{\delta x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \quad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}}; \quad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\frac{g}{l} & 0\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix} + \begin{bmatrix}0\\ \frac{1}{ml}\end{bmatrix}\delta u$$

Remark: There is also another equilibrium point at $x_2=u=0$, $x_1=\pi$

Example 3 *Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$*



$$M\frac{d^2y}{dt^2}=Mg-\frac{i^2(t)}{y}$$

$$e(t)=R_1 i+L\frac{di}{dt}$$

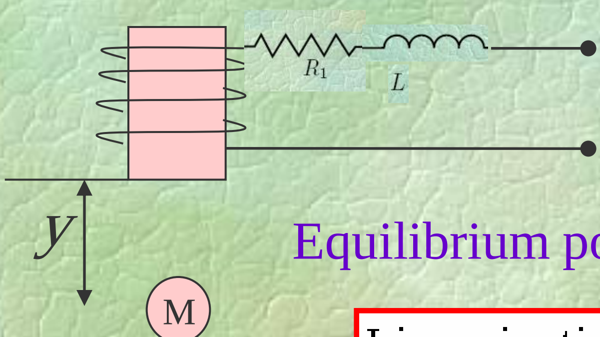
$$x_1=y,\ x_2=\dot{y},\ x_3=i$$

$$\dot{x}_1=x_2 \qquad \dot{x}_2=g-\frac{1}{M}\frac{{x_3}^2}{x_1} \qquad \dot{x}_3=-\frac{R_1}{L}x_3+\frac{e(t)}{L}$$

Equilibrium point:

$$x_{1Q}=y_0 \qquad x_{2Q}=0 \qquad x_{3Q}=\mathrm{i}_Q=\sqrt{Mgy_0} \qquad e_Q=R_1\sqrt{Mgy_0}$$

Example 3 *Suppose electromagnetic force is i^2/y and find linearzed model around $y=y_0$*



$$\dot{x}_1 = x_2 \qquad \dot{x}_2 = g - \frac{1}{M}\frac{x_3^{\;2}}{x_1} \qquad \dot{x}_3 = -\frac{R_1}{L}x_3 + \frac{e(t)}{L}$$

Equilibrium point: $\left(x_{1Q}, x_{2Q}, x_{3Q}, e_Q\right) = \left(y_0, 0, \sqrt{Mgy_0}, R_1\sqrt{Mgy_0}\right)$

Linearization procedure

$$\delta\dot{x} = A\,\delta x + B\,\delta u \qquad A = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad B = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta y = C\,\delta x + D\,\delta u \qquad C = \left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_Q\\u=u_Q}} ; \qquad D = \left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_Q\\u=u_Q}}$$

$$\delta\ddot{x}_1 = \delta x_2$$

$$\delta\ddot{x}_2 = \frac{g}{y_0}\delta x_1 - 2\sqrt{\frac{g}{My_0}}\,\delta x_3$$

$$\delta\ddot{x}_3 = -\frac{R_1}{L}\delta x_3 + \frac{\delta e(t)}{L}$$

$$\begin{bmatrix}\delta\ddot{x}_1\\ \delta\ddot{x}_2\\ \delta\ddot{x}_3\end{bmatrix} = \begin{bmatrix}0 & 1 & 0\\ \frac{g}{y_0} & 0 & -2\sqrt{\frac{g}{My_0}}\\ 0 & 0 & -\frac{R_1}{L}\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{bmatrix} + \begin{bmatrix}0\\ 0\\ \frac{1}{L}\end{bmatrix}\delta e(t)$$

Example 4 Consider the following nonlinear system. Suppose $u(t)=0$ and initial condition is $x_{10}=x_{20}=1$. Find the linearized system around response of system.

$$\dot{x}_1(t)=\frac{-1}{x_2(t)^2}$$

$$\dot{x}_2(t)=u(t)x_1(t)$$

$$\dot{x}_2(t)=0.x_1(t)=0 \quad \Rightarrow \quad x_2(t)=a=1$$

$$\dot{x}_1(t)=-1 \quad \Rightarrow \quad x_1(t)=-t+b=-t+1$$

Linearization procedure

$$\dot{\delta}x=A\delta x+B\delta u \qquad A=\left.\frac{\partial f}{\partial x}\right|_{\substack{x=x_g\\u=u_g}}; \qquad B=\left.\frac{\partial f}{\partial u}\right|_{\substack{x=x_g\\u=u_g}}$$

$$\delta y=C\delta x+D\delta u \qquad C=\left.\frac{\partial g}{\partial x}\right|_{\substack{x=x_g\\u=u_g}}; \qquad D=\left.\frac{\partial g}{\partial u}\right|_{\substack{x=x_g\\u=u_g}}$$

$$\begin{bmatrix}\delta\dot{x}_1\\ \delta\dot{x}_2\end{bmatrix}=\begin{bmatrix}0 & 2\\ 0 & 0\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\end{bmatrix}+\begin{bmatrix}0\\ 1-t\end{bmatrix}\delta u(t)$$

Linear model for time delay

$$e^{-\tau s} = \frac{1}{e^{\tau s}} = \frac{1}{1+\tau s+\frac{\tau^2}{2}s^2+\frac{\tau^3}{6}s^3+........} \cong \frac{1}{1+\tau s+\frac{\tau^2}{2}s^2}$$

$$e^{-\tau s} = \frac{e^{-\frac{\tau}{2}s}}{e^{\frac{\tau}{2}s}} \cong \frac{1-\frac{\tau}{2}s}{1+\frac{\tau}{2}s}$$

Pade approximation

Example 5(Inverted pendulum)

مثال ۵ پاندول معکوس

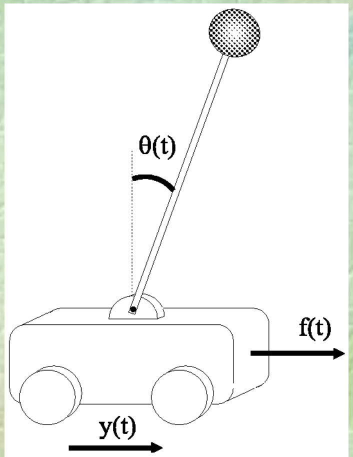


Figure 3.5: *Inverted pendulum*

In Figure 3.5, we have used the following notation:

$y(t)$	-	distance from some reference point
$\theta(t)$	-	angle of pendulum
M	-	mass of cart
m	-	mass of pendulum (assumed concentrated at tip)
l	-	length of pendulum
$f(t)$	-	forces applied to pendulum

Example of an Inverted Pendulum

Nonlinear model

Application of Newtonian physics to this system leads to the following model:

$$\ddot{y} = \frac{1}{\lambda_m + \sin^2\theta(t)}\left[\frac{f(t)}{m} + \dot{\theta}^2(t)\ell\sin\theta(t) - g\cos\theta(t)\sin\theta(t)\right]$$

$$\ddot{\theta} = \frac{1}{\ell\lambda_m + \sin^2\theta(t)}\left[-\frac{f(t)}{m}\cos\theta(t) + \dot{\theta}^2(t)\ell\sin\theta(t)\cos\theta(t) + (1-\lambda_m)g\sin\theta(t)\right]$$

where $\lambda_m = (M/m)$

Linear model

This is a linear state space model in which A, B and C are:

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{(M+m)g}{M\ell} & 0 \end{bmatrix}; \quad \mathbf{B} = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ -\frac{1}{M\ell} \end{bmatrix}; \quad \mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}$$

TF model properties خواص مدل تابع انتقال

1- It is available just for linear time invariant systems. (LTI)

2- It is derived by zero initial condition.

3- It just shows the relation between input and output so it may lose some information.

4- It is just dependent on the structure of system and it is independent to the input value and type.

5- It can be used to show delay systems but SS can not.

Poles of Transfer function قطب تابع انتقال

Transfer function

$$G(s)=\frac{n(s)}{d(s)}$$

Pole: *s=p* is a value of s that the absolute value of TF is infinite.

How to find it? Let *d(s)=0* and find the roots

Why is it important? It shows the behavior of the system.

Why?????

Example 6

Poles and their physical meaning

قطبها و خواص فیزیکی آنها

Transfer function

$$G(s)=\frac{s+6}{s^2+5s+6}$$

$S^2+5s+6=0$ → $p_1=-2,\ p_2=-3$

MATLAB roots([1 5 6])

$$\text{Step Response of } c(s)=\frac{s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3}$$

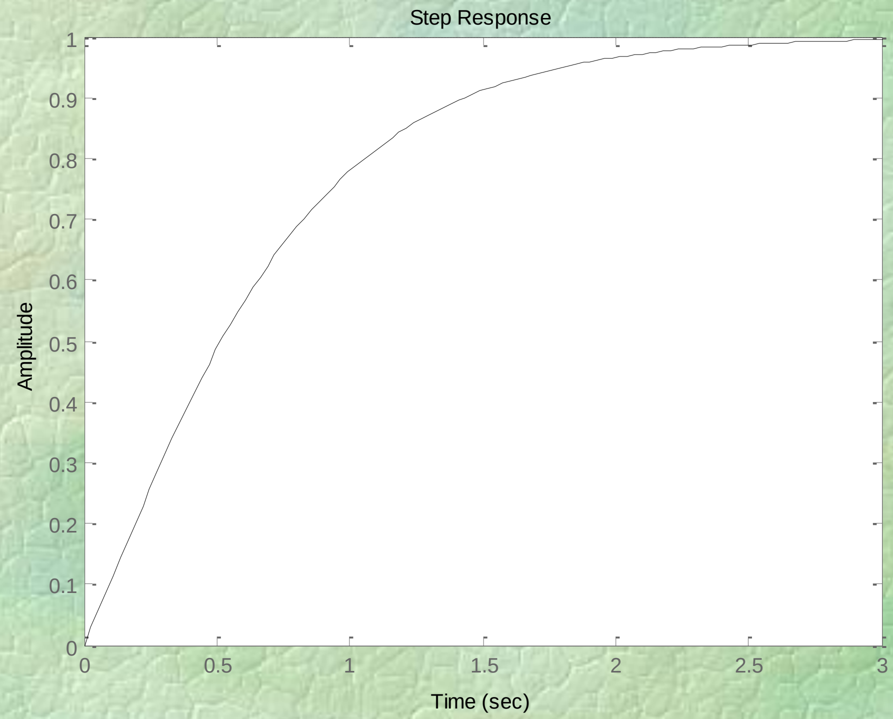
$$c(t)=1u(t)-2e^{-2t}u(t)+1e^{-3t}u(t)$$

$p_1=-2,\ p_2=-3$

Example 6 (Continue)

Transfer function

$$G(s)=\frac{s+6}{s^2+5s+6}$$



MATLAB step([1 6],[1 5 6])

Zeros of Transfer function صفر تابع انتقال

Transfer function

$$G(s) = \frac{n(s)}{d(s)}$$

Zero: *s=z* is a value of s that the absolute value of TF is zero.

Who to find it? Let *n(s)=0* and find the roots

Why is it important? It also shows the behavior of system.

Why?????

Example 7 Zeros and their physical meaning

صفرها و خواص فیزیکی آنها

Transfer functions

$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad G_2(s)=\frac{10s+6}{s^2+5s+6}$$

z_1= -6 $\qquad$ z_2=-0.6

$$c_1(s)=\frac{s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{-2}{s+2}+\frac{1}{s+3} \qquad c_1(t)=\left(1-2e^{-2t}+1e^{-3t}\right)u(t)$$

$$c_2(s)=\frac{10s+6}{s(s^2+5s+6)}=\frac{1}{s}+\frac{7}{s+2}+\frac{-8}{s+3} \qquad c_2(t)=\left(1+7e^{-2t}-8e^{-3t}\right)u(t)$$

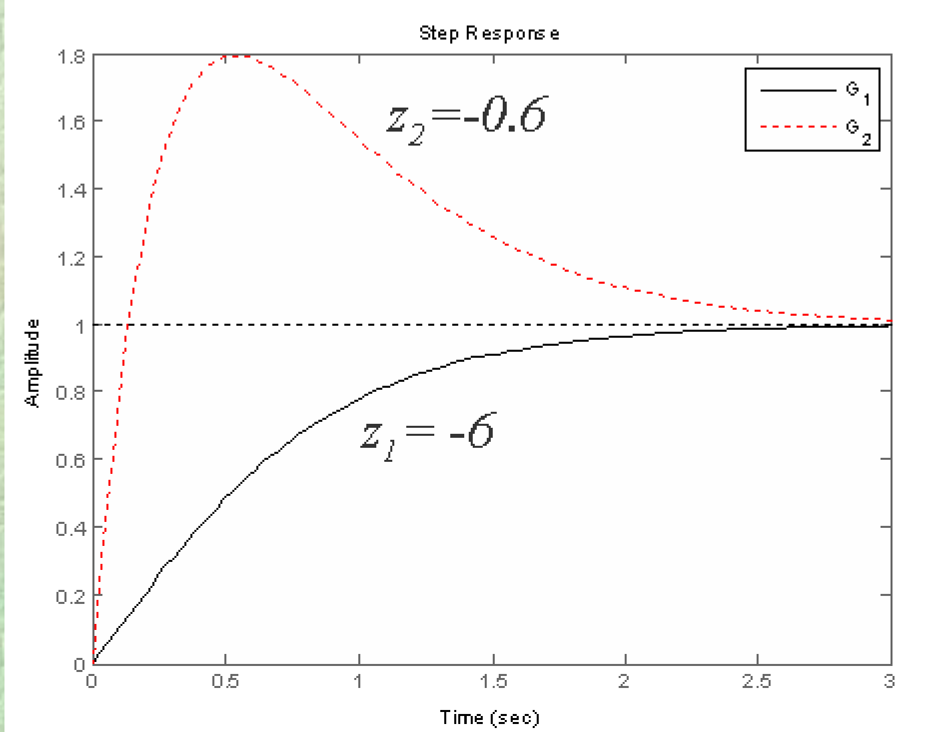
Zeros and their physical meaning

Example 7

صفرها و خواص فیریکی آنها

Transfer function

$$G_1(s)=\frac{s+6}{s^2+5s+6} \qquad G_2(s)=\frac{10s+6}{s^2+5s+6}$$



MATLAB

step([1 6],[1 5 6]) ;hold on;step([10 6],[1 5 6])

Example 8

Zeros and its physical meaning

صفرها و خواص فيريكى آنها

$$\ddot{y}+5\dot{y}+6y=\dot{u}+u$$

Transfer function of the system is:

$$\frac{y(s)}{u(s)}=\frac{s+1}{s^2+5s+6}$$

It has a zero at *z=-1*

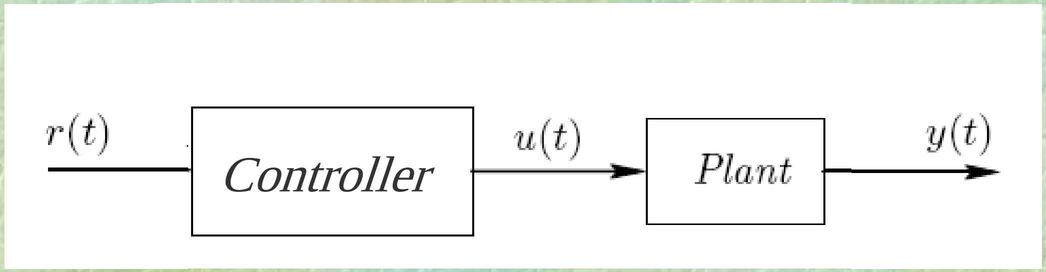
What does it mean?

$u=e^{-1t}$ and suitable y_0 and y'_0 leads to $y(t)=\underline{0}$

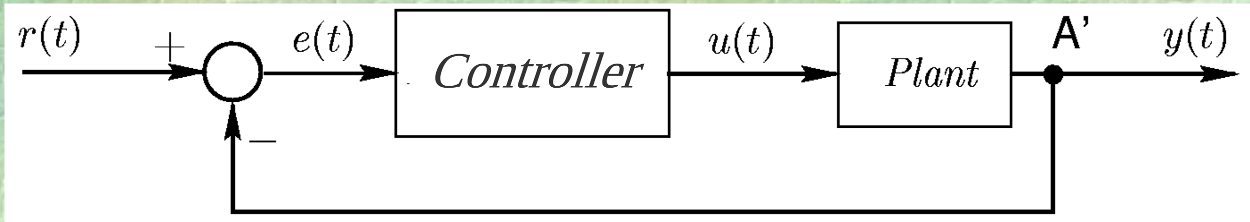
Open loop and closed loop systems

سیستمهای حلقه باز و حلقه بسته

Open loop



Closed loop



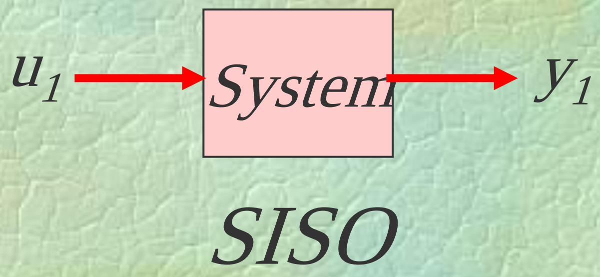
When is the open loop representation applicable?

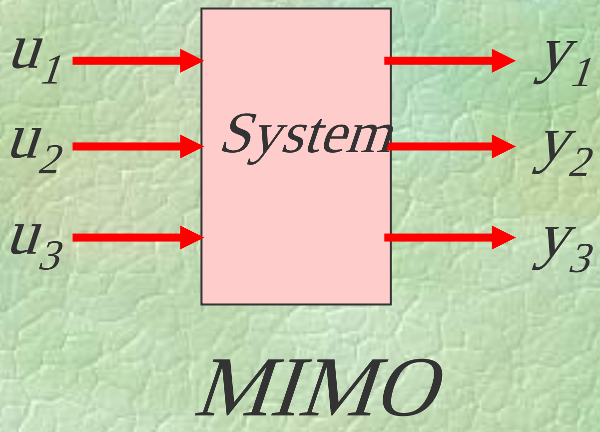
در چه شرایطی کنترل حلقه باز مناسب است؟

- The model on which the design of the controller has been based is a very good representation of the plant,
- The model must be stable, and
- Disturbances and initial conditions are negligible.

SISO and MIMO

سیستم تک ورودی تک خروجی و چند ورودی چند خروجی





Sensitivity حساسیت

Sensitivity is defined as:

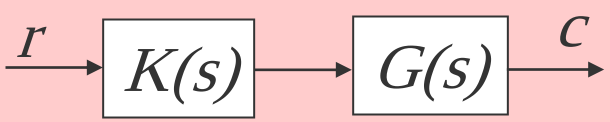
$$S = \frac{\partial M / M}{\partial G / G} = \frac{\partial M}{\partial G}\frac{G}{M}$$

M is the system + controller transfer function

G is the system transfer function

Sensitivity

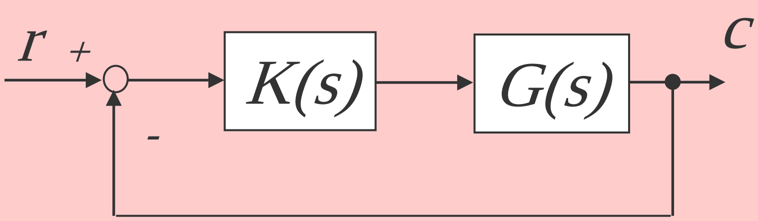
حساسیت



Open loop system

$$M = G(s)K(s)$$

$$S = \frac{\partial M}{\partial G}\frac{G}{M} = K(s)\frac{G(s)}{K(s)G(s)} = 1$$



Closed loop system

$$M(s) = \frac{K(s)G(s)}{1+K(s)G(s)}$$

$$S = \frac{\partial M}{\partial G}\frac{G}{M} = \frac{1}{1+K(s)G(s)}$$

Feedback properties ویژگیهای فیدبک

Feedback can effect:

a) system gain

b) system stability

c) system sensitivity

d) noise and disturbance

Exercises

5-1 Find the poles and zeros of following system.

$$G(s)=\frac{s^2+4}{S^3+3s^2+3s+3}$$

5-2 Is it possible to apply a nonzero input to the following system for t>0, but the output be zero for t>0? Show it.

$$y''+5y'+6y=2u'+u$$

5-3 Find the step response of following system.

$$G(s)=\frac{s^2+4}{s^3+3s^2+3s+3}$$

5-4 Find the step response of following system for a=1,3,6 and 9.

$$G(s)=\frac{as+2}{s^2+2s+2}$$

5-5 Find the linear model of system in example 2 around x2=u=0, x1=π