LINEAR CONTROL SYSTEMS

Ali Karimpour
Associate Professor
Ferdowsi University of Mashhad

Lecture 6

Topics to be covered include:

- State space representation.
  - State transition matrix.
  - Calculation of state transition matrix for time varying system.
  - Significance of state transition matrix and its property.
- State transition equation.
- State transition equation determined by state diagram.
- Eigenvalues of the matrix A in state space form and poles of transfer function
- Similarity transformation.

State Space Models مدل فضای حالت

General form of state space model

$$\frac{dx}{dt} = f(x(t), u(t), t)$$
$$y(t) = g(x(t), u(t), t)$$

LTI systems

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

If initial condition and input are defined, then x(t), y(t) ?

اگر شرائط اولیه و ورودی مشخص باشد مقدار y(t) و x(t) چند است؟

Start with homogeneous equation با معادله همگن شروع کنید.

$$\dot{x} = Ax \qquad x(t_0) = x_0$$

State transition matrix ماتریس انتقال حالت

$$\dot{x} = Ax \qquad x(t_0 = 0) = x(0)$$

$$x(t) = \phi(t)x(0)$$

$\phi(t)$: is state transition matrix ماتریس انتقال حالت

If t_0 is not zero

$$x(t) = \phi(t - t_0)x(t_0)$$

Determination of state transition matrix

تعیین ماتریس انتقال حالت

Method I $\dot{x} = Ax$ $x(t_0) = x_0$ روش اول

$$sx(s) - x_0 = Ax(s) \rightarrow x(s) = (sI - A)^{-1} x_0$$

$$x(t) = L^{-1}\left((sI - A)^{-1}\right) x_0$$

$x(t) = \phi(t) x_0$

$$\phi(t) = L^{-1}\left((sI - A)^{-1}\right)$$

Determination of state transition matrix

تعیین ماتریس انتقال حالت

Method II $\dot{x} = Ax$ $x(t_0) = x_0$ روش دوم

$$x(t) = e^{At} x_0$$

$$x(t) = \phi(t) x_0$$

$$\phi(t) = e^{At}$$

$$e^{At} = I + At + \frac{1}{1.2} A^2 t^2 + \frac{1}{1.2.3} A^3 t^3 + \dots = \mathbf{I} + \sum_{i=1}^{\infty} \frac{1}{i!} \mathbf{A}^i t^i$$

Determination of state transition matrix

تعیین ماتریس انتقال حالت

Method III

روش سوم

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t) \\ \varphi_{21}(t) \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} \varphi_{11}(t) & \varphi_{12}(t) \\ \varphi_{21}(t) & \varphi_{22}(t) \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} \varphi_{12}(t) \\ \varphi_{22}(t) \end{bmatrix}$$

So i^{th} column of Φ is the response of system to $\begin{bmatrix} 0 \\ 0 \\ . \\ . \\ 1 \\ . \\ . \\ 0 \end{bmatrix}$ ← ***i^{th} position***

Example 1 State transition matrix and *x(t)*

مثال ۱: ماتریس انتقال حالت و *x(t)* را بیابید.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x \qquad x(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$x(t) = L^{-1}\left((sI - A)^{-1}\right)x_0$$

$$\phi(t) = L^{-1}\left((sI - A)^{-1}\right) = L^{-1}\left(\begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1}\right) = L^{-1}\left(\begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix}\right)$$

$$\phi(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3e^{-2t} - e^{-3t} \\ e^{-3t} \end{bmatrix}$$

Property of state transition matrix for LTI systems

خواص ماتریس انتقال حالت در سیستمهای LTI

1- $\Phi(0) = I$

2- $\Phi^{-1}(t)= \Phi(-t)$

3- $\Phi(t_2-t_1)\Phi(t_1-t_0) = \Phi(t_2-t_0)$

4- $\Phi^k(t) = \Phi(kt)$

State transition matrix for time varying systems

ماتریس انتقال حالت برای سیستمهای متغیر با زمان

$$\dot{x} = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix} x$$

$$\dot{x}_1(t) = 0$$
$$\dot{x}_2(t) = t x_1(t)$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 1 \\ 0.5t^2 \end{bmatrix}$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \rightarrow x(t) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} 1 & 0 \\ 0.5t^2 & 1 \end{bmatrix}$$

State transition equation

معادله انتقال حالت

$$\dot{x} = Ax + Bu \qquad x(t_0) = x_0$$

$$sx(s) - x_0 = Ax(s) + Bu(s)$$

$$x(s) = (sI - A)^{-1}x_0 + (sI - A)^{-1}Bu(s)$$

$$x(t) = \phi(t)x_0 + \int_0^t \phi(t-\tau)Bu(\tau)d\tau$$

Convolution integral

State transition equation

$$x(t) = e^{\mathbf{A}(t-t_o)}x_o + \int_{t_o}^{t} e^{\mathbf{A}(t-\tau)}\mathbf{B}u(\tau)d\tau$$

Example 2 Find state transition equation for unit step

مثال ۲: معادله انتقال حالت را با فرض اعمال پله واحد بیابید.

$$\dot{x}=\begin{bmatrix}-2 & 1\\ 0 & -3\end{bmatrix}x+\begin{bmatrix}0\\1\end{bmatrix}u \qquad x(0)=\begin{bmatrix}x_1(0)\\x_2(0)\end{bmatrix}$$

$$x(t)=L^{-1}\left((sI-A)^{-1}\right)x_0$$

$$\phi(t)=L^{-1}\left((sI-A)^{-1}\right)=L^{-1}\left(\begin{bmatrix}s+2 & -1\\ 0 & s+3\end{bmatrix}^{-1}\right)=L^{-1}\left(\begin{bmatrix}\frac{1}{s+2} & \frac{1}{(s+2)(s+3)}\\ 0 & \frac{1}{s+3}\end{bmatrix}\right)$$

$$\phi(t)=\begin{bmatrix}e^{-2t} & e^{-2t}-e^{-3t}\\ 0 & e^{-3t}\end{bmatrix}$$

Example 2 Find state transition equation for unit step

مثال ۲: معادله انتقال حالت را با فرض اعمال پله واحد بیابید.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$

$$x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

$$\phi(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix}$$

$$x(t) = \phi(t)x_0 + \int_0^t \phi(t-\tau)bu(\tau)d\tau$$

$$x(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} & e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ 0 & e^{-3(t-\tau)} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(\tau)d\tau$$

$$x(t) = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \int_0^t \begin{bmatrix} e^{-2(t-\tau)} - e^{-3(t-\tau)} \\ e^{-3(t-\tau)} \end{bmatrix} d\tau = \begin{bmatrix} \cdots\cdots \\ \cdots\cdots \end{bmatrix}$$

Example 3 Find $x_1(s)$ and $x_2(s)$

مثال ۳: معادله $x_1(s)$ و $x_2(s)$ را بیابید.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \qquad x(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$$

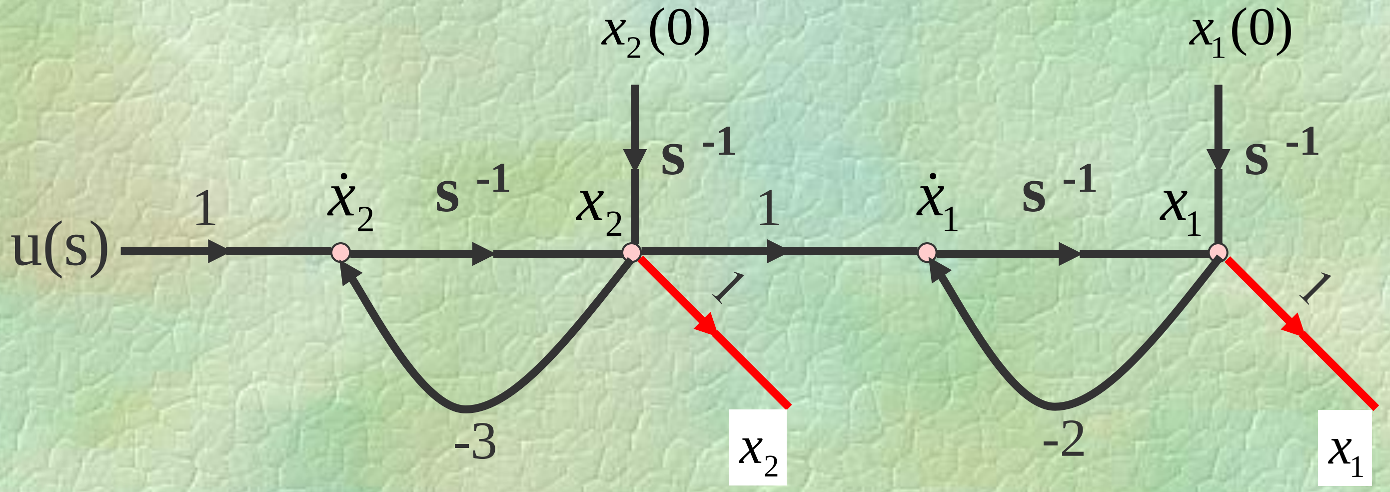
$$x(s) = (sI - A)^{-1} x(0) + (sI - A)^{-1} Bu(s)$$

$$x(s) = \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1} x(0) + \begin{bmatrix} s+2 & -1 \\ 0 & s+3 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(s)$$

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{s+2} & \frac{1}{(s+2)(s+3)} \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} \frac{1}{(s+2)(s+3)} \\ \frac{1}{s+3} \end{bmatrix} u(s) = \begin{bmatrix} \frac{1}{s+2} x_1(0) + \frac{1}{(s+2)(s+3)} x_2(0) + \frac{1}{(s+2)(s+3)} u(s) \\ \frac{1}{s+3} x_2(0) + \frac{1}{s+3} u(s) \end{bmatrix}$$

Example 4: Derive $x_1(s)$ by using state diagram.

$$\dot{x} = \begin{bmatrix} -2 & 1 \\ 0 & -3 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u$$



$$x_1(s) = \frac{1}{s+2} x_1(0) + \frac{1}{(s+2)(s+3)} x_2(0) + \frac{1}{(s+2)(s+3)} u(s)$$

Relation between SS model and TF model.

ارتباط بین معادلات فضای حالت و تابع انتقال

$$\frac{dx(t)}{dt} = \mathbf{A}x(t) + \mathbf{B}u(t)$$
$$y(t) = \mathbf{C}x(t) + \mathbf{D}u(t)$$

$$Y(s) = \mathbf{G}(s)U(s)$$
$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{I} - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D}$$

$$G(s) = C\frac{adj(sI - A)}{|sI - A|}B + D$$

Poles of G(s) ?   Eigenvalues of A (modes) ?

ارتباط بین قطبهای تابع انتقال و مقادیر ویژه A ؟

Poles of G(s) $\subseteq$ Eigenvalues of A

Example 5: Find the poles of transfer function and the eigenvalues of A

$$\dot{x}=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -6 & -11 & -6\end{bmatrix}x+\begin{bmatrix}0\\0\\1\end{bmatrix}u$$

$$y=[1\quad 1\quad 0]x$$

مثال ۵: مطلوبست قطبهای تابع انتقال و مقادیر ویژه A

$$|SI-A|=\begin{vmatrix}s & -1 & 0\\ 0 & s & -1\\ 6 & 11 & s+6\end{vmatrix}=s^3+6s^2+11s+6=(s+1)(s+2)(s+3)$$

MATLAB

eig(A)

$$g(s)=d+c(sI-A)^{-1}b=\frac{s+1}{s^3+6s^2+11s+6}=\frac{s+1}{(s+1)(s+2)(s+3)}=\frac{1}{(s+2)(s+3)}$$

MATLAB

[num,den]=ss2tf(A,b,c,0), g=tf(num,den) , g=minreal(g)

Example5: Continue

مثال۵: ادامه

$$\dot{x} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & -11 & -6 \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u$$

$$y = [1 \quad 1 \quad 0] x$$

$$G(s) = \frac{1}{(s+2)(s+3)}$$

$\lambda_1 = -3$

$\lambda_2 = -2$

$\lambda_3 = -1$

Eigenvalues of *A*

مقادیر ویژه *A*

$p_1 = -3$

$p_2 = -2$

Poles of *system*

قطبهای سیستم

What happened to -1 ??!!??

Change of variables (Similarity transformation)

تغییر متغیرها (تبدیلات همانندی)

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Let $A \ is \ 3\times 3$ $\quad b \ is \ 3\times 1$

$c \ is \ 1\times 3$ $\quad d \ is \ 1\times 1$

$$\begin{aligned} w_1 &= x_1 - x_3 \\ w_2 &= x_1 + 2x_2 \\ w_3 &= x_2 + x_3 \end{aligned} \Rightarrow w = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix} x = Px \Rightarrow x = P^{-1}w = \begin{bmatrix} 1 & 0 & -1 \\ 1 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}^{-1} w$$

$$\begin{aligned} \dot{x} &= Ax + bu \\ y &= cx + du \end{aligned} \Rightarrow \begin{aligned} P^{-1}\dot{w} &= AP^{-1}w + bu \\ y &= cP^{-1}w + du \end{aligned} \Rightarrow \begin{aligned} \dot{w} &= PAP^{-1}w + Pbu \\ y &= cP^{-1}w + du \end{aligned}$$

$\hat{A} \quad \hat{b} \quad \hat{c} \quad \hat{d}$

$$\dot{w} = \hat{A}w + \hat{b}u$$
$$y = \hat{c}w + \hat{d}u$$

Example 6: A similarity transformation example

مثال ۶: یک مثال از تبدیل همانندی

$$\dot{x}=\begin{bmatrix}-3 & -6 & -4\\ 1 & 2 & 2\\ -1 & -6 & -6\end{bmatrix}x+\begin{bmatrix}6\\ -3\\ 4\end{bmatrix}u$$

$$y=[2 \quad 2 \quad -1]x$$

$\lambda_1=-2$

$\lambda_2=-4$

$\lambda_3=-1$

Eigenvalues of A

مقادیر ویژه A

$$P^{-1}=\begin{bmatrix}-2 & -1 & 2\\ 1 & 1 & -1\\ -1 & -1 & 2\end{bmatrix}$$

$$\hat{A}=PAP^{-1}=\begin{bmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -4\end{bmatrix}$$

$$\hat{b}=Pb=\begin{bmatrix}-4.9\\ 0\\ 3\end{bmatrix}$$

$$\hat{c}=cP^{-1}=[-0.4 \quad 0.6 \quad 0]$$

$$\hat{d}=d=0$$

Eigenvector corresponding to -2

$$\dot{w}=\begin{bmatrix}-2 & 0 & 0\\ 0 & -1 & 0\\ 0 & 0 & -4\end{bmatrix}w+\begin{bmatrix}-4.9\\ 0\\ 3\end{bmatrix}u$$

$$y=[-0.4 \quad 0.6 \quad 0]w$$

$\lambda_1=-2$

$\lambda_2=-4$

$\lambda_3=-1$

Eigenvalues of $\hat{A}$

مقادیر ویژه $\hat{A}$

New system is very simpler than older

Similarity transformation and eigenvalues

تغییر متغیرها و مقادیر ویژه

$$\dot{x} = Ax + bu$$
$$y = cx + du$$

Eigenvalues of A $\Rightarrow$ $|sI - A| = 0$

مقادیر ویژه A

$$\dot{z} = \hat{A}z + \hat{b}u$$
$$y = \hat{c}z + \hat{d}u$$

Eigenvalues of $\hat{A}$ $\Rightarrow$ $|sI - \hat{A}| = 0$

مقادیر ویژه $\hat{A}$

$$|sI - \hat{A}| = |sI - PAP^{-1}| = |PsP^{-1} - PAP^{-1}| = |P||sI - A||P^{-1}| = |sI - A|$$

$$\Rightarrow \quad |sI - \hat{A}| = |sI - A|$$

Similarity transformation does not affect the eigenvalues of the system

Exercise

1- Find the state diagram of the following system

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

2- Find the transfer function of the following system.

a) By use of the formula between the two representation.

b) By use of state diagram

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x$$

3- Find *y(t)* for initial condition $[1\ 3\ \ -1]^T$ and the unit step as input.

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix} x$$

Exercise (cont.)

4- The state transition matrix and state transition equation of following system.

$$\dot{X}(t)=\begin{bmatrix}-1 & 3\\ 0 & -4\end{bmatrix}X(t)+\begin{bmatrix}1\\ 2\end{bmatrix}r(t)$$

Answer is:

$$\phi(t)=L^{-1}\left((sI-A)^{-1}\right)=L^{-1}\left(\begin{bmatrix}s+1 & -3\\ 0 & s+4\end{bmatrix}^{-1}\right)=L^{-1}\begin{bmatrix}\frac{1}{s+1} & \frac{3}{s^2+5s+4}\\ 0 & \frac{1}{s+4}\end{bmatrix}=\begin{bmatrix}e^{-t} & e^{-t}-e^{-4t}\\ 0 & e^{-4t}\end{bmatrix}$$

$$X(t)=\begin{bmatrix}e^{-t} & e^{-t}-e^{-4t}\\ 0 & e^{-4t}\end{bmatrix}X(0)+\int_0^t\begin{bmatrix}e^{-(t-\tau)} & e^{-(t-\tau)}-e^{-4(t-\tau)}\\ 0 & e^{-4(t-\tau)}\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}R(\tau)dt$$

5- Suppose $x_1(0)=1$, $x_2(0)=3$ and $x_3(0)=2$ and $r(t)=u(t)$. Find $x(t)$ and $c(t)$ for t≥0

$$\dot{X}(t)=\begin{bmatrix}-1 & 4 & 0\\ 2 & -2 & 1\\ 0 & 0 & -3\end{bmatrix}X(t)+\begin{bmatrix}1\\ 1\\ 0\end{bmatrix}r(t) \qquad c(t)=[1 \quad 0 \quad 0]X(t)$$