LINEAR CONTROL SYSTEMS

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Lecture 9

Stability analysis

Topics to be covered include:

- Stability of linear control systems.
  - Bounded input bounded output stability (BIBO).
  - Zero input stability.
- Stability of linear control systems through Routh Hurwitz criterion.

Stability analysis تجزیه تحلیل پایداری

The response of linear systems can always be decomposed as the zero-state response and zero-input response. We study

1. Input output stability of LTI system is called BIBO (bounded-input bounded-output) stability (the zero-state response)

2. Internal stability of LTI system is called Asymptotic stability (the zero-input response)

پاسخ سیستمهای خطی را می توان بصورت جمع پاسخ حالت صفر و پاسخ ورودی صفر بیان نمود.

۱- پایداری ورودی خروجی سیستمهای خطی پایداری BIBO (ورودی کراندار خروجی کراندار) نامیده می شود. (پاسخ حالت صفر)

۲- پایداری داخلی سیستمهای خطی پایداری مجانبی نامیده می شود. (پاسخ ورودی صفر)

Input output stability of LTI system

پایداری ورودی خروجی سیستمهای LTI

Consider a SISO linear time-invariant system, then the output can be described by

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \qquad (I)$$

where *g(t)* is the impulse response of the system and system is relaxed at *t=0*.

در سیستم تک ورودی تک خروجی خطی غیر متغیر با زمان (LTI) خروجی را می توان بصورت

$$y(t) = \int_0^t g(t-\tau)u(\tau)d\tau = \int_0^t g(\tau)u(t-\tau)d\tau \qquad (I)$$

نمایش داد که *g(t)* پاسخ ضربه بوده و سیستم در *t=0* آرام است.

Input output stability of LTI system

پایداری ورودی خروجی سیستمهای LTI

Definition: A system is said to be BIBO stable (bounded-input bounded-output) if every bounded input excited a bounded output. This stability is defined for zero-state response and is applicable only if the system is initially relaxed.

تعریف: یک سیستم را پایدار BIBO گویند اگر هر ورودی محدود خروجی محدود را تولید کند. این پایداری برای پاسخ حالت صفر تعریف شده و سیستم در ابتدا آرام است.

Input output stability of LTI system

پایداری ورودی خروجی سیستمهای LTI

Theorem: A SISO system described by (I) is BIBO stable if and only if g(t) is absolutely integrable in [0,∞), or

$$\int_0^\infty |g(t)|dt \le M < \infty$$

For some constant M.

قضیه: یک سیستم SISO توصیف شده با معادلات (I) را پایدار BIBO گویند اگر و فقط اگر قدر مطلق g(t) در بازه [0,∞) انتگرال پذیر باشد یا

$$\int_0^\infty |g(t)|dt \le M < \infty$$

M عدد ثابت می باشد.

Input output stability of LTI system

پایداری ورودی خروجی سیستمهای LTI

Proof: g(t) is absolutely integrable ⇨ system is BIBO

Let $u(t)$ be bounded so $|u(t)| < u_m$ then

$$|y(t)| = \left|\int_0^{\infty} g(\tau)u(t-\tau)d\tau\right| \le \int_0^{\infty} |g(\tau)||u(t-\tau)|d\tau \le u_m \int_0^{\infty} |g(\tau)|d\tau \le u_m M$$

So the output is bounded.

Now: System is BIBO stable ⇨ g(t) is absolutely integrable

If $g(t)$ is not absolutely integrable, then there exists t_1 such that:

$$\int_0^{t_1} |g(\tau)|d\tau = \infty$$

Let us choose

$$u(t_1-\tau) = \begin{cases} 1 & \text{if } g(\tau) \ge 0 \\ -1 & \text{if } g(\tau) < 0 \end{cases}$$

$$y(t_1) = \int_0^{t_1} g(\tau)u(t_1-\tau)d\tau = \int_0^{t_1} |g(\tau)|d\tau = \infty$$ **So it is not BIBO**

Input output stability of LTI system

پایداری ورودی خروجی سیستمهای LTI

Theorem: A SISO system with proper rational transfer function $g(s)$ is BIBO stable if and only if every pole of $g(s)$ has negative real part.

قضیه: یک سیستم SISO با تابع انتقال مناسب گویای $g(s)$ را پایدار BIBO گویند اگر و فقط اگر هر قطب $g(s)$ دارای قسمت حقیقی منفی باشد.

If $\hat{g}(s)$ has pole p_i with multiplicity m_i, then its partial fraction expansion contains the factors

$$\frac{1}{s-p_i},\quad \frac{1}{(s-p_i)^2},\quad \cdots,\quad \frac{1}{(s-p_i)^{m_i}}$$

Thus the inverse Laplace transform of $\hat{g}(s)$ or the impulse response contains the factors

$$e^{p_i t},\quad te^{p_i t},\quad \cdots,\quad t^{m_i-1}e^{p_i t}$$

It is straightforward to verify that every such term is absolutely integrable if and only if p_i has a negative real part.

Internal stability

پایداری داخلی

The BIBO stability is defined for the zero-state response. Now we study the stability of the zero-input response.

Definition: The zero-input response of $\dot{x} = Ax$ is stable in the sense of Lyapunov if every finite initial state x_0 excites a bounded response. In addition if the response approaches to zero then it is asymptotically stable.

تعریف: پاسخ ورودی صفر سیستم $\dot{x} = Ax$ را به مفهوم لیاپانوف پایدار گویند اگر هر حالت اولیه محدود x_0 پاسخ محدودی را بوجود آورد. علاوه بر این اگر پاسخ به صفر میل کند پایداری مجانبی حاصل می شود.

Internal stability

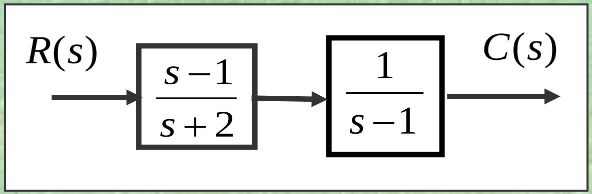
پایداری داخلی

Theorem: The equation $\dot{x} = Ax$ is asymptotically stable if and only if all eigenvalues of A have negative real parts.

قضیه: معادله $\dot{x} = Ax$ پایدار مجانبی است اگر و فقط اگر تمام مقادیر ویژه A دارای قسمت حقیقی منفی باشد.

Relation between BIBO stability and asymptotic stability?

Example 1: Discuss the stability of the system .



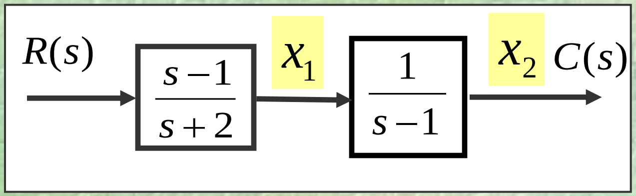
BIBO stability:

$$G(s)=\frac{C(s)}{R(s)}=\frac{1}{s+2}$$

There is no RHP root , so system is BIBO stable.

Internal stability:

For internal stability we need state-space model so we have:



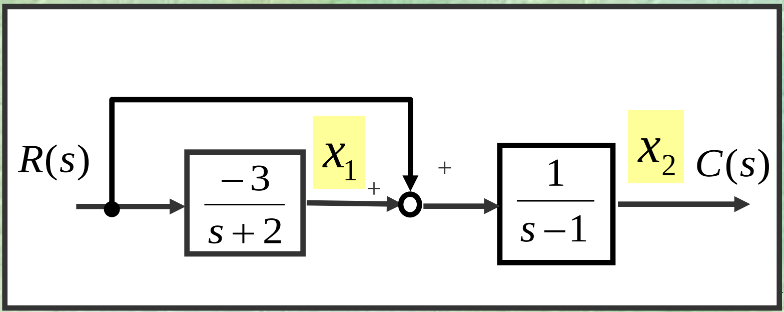
$$x_2(s)=\frac{1}{s-1}x_1(s) \longrightarrow \dot{x}_2=x_1+x_2$$

$$x_1(s)=\frac{s-1}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1+\dot{r}-r$$

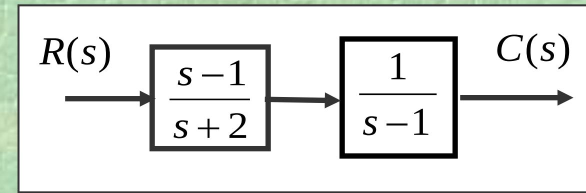
?

$$x_2(s)=\frac{1}{s-1}\left(x_1(s)+R(s)\right) \longrightarrow \dot{x}_2=x_1+x_2+r$$

$$x_1(s)=\frac{-3}{s+2}R(s) \longrightarrow \dot{x}_1=-2x_1-3r$$



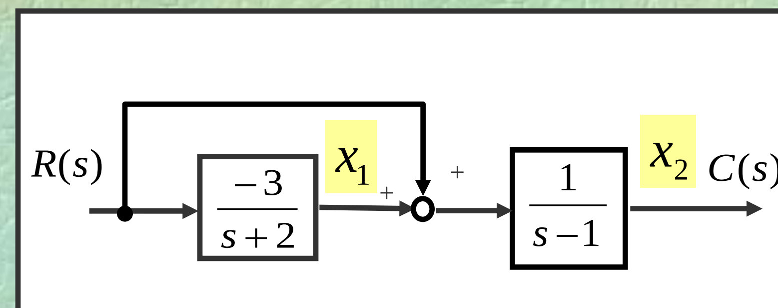
Example 1: Discuss the stability of the system .



BIBO stability: There is no RHP root , so system is BIBO stable.

Internal stability:

For internal stability we need state-space model so we have:



$\dot{x}_1 = -2x_1 - 3r$ $\qquad$ $\dot{x}_2 = x_1 + x_2 + r$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} -3 \\ 1 \end{bmatrix} r$$

$$c = \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\Rightarrow$

$$|sI - A| = \begin{vmatrix} s+2 & 0 \\ -1 & s-1 \end{vmatrix} = (s-1)(s+2)$$

$$\rightarrow \lambda_1 = +1, \lambda_2 = -2$$

The system is not internally stable (neither asymptotic nor Lyapunov stable).

Very important note: If RHP poles and zeros between different part of system omitted then the system is internally unstable although it may be BIBO stable.

Review

مرور

How can we check BIBO stability?

$G(s)=\frac{n(s)}{d(s)}$ → Let $d(s)=0$ → Find poles $p_1, p_2, ..., p_n$

→ If $p_1, p_2, ..., p_n \in$ LHP → System is BIBO stable

How can we check asymptotic stability?

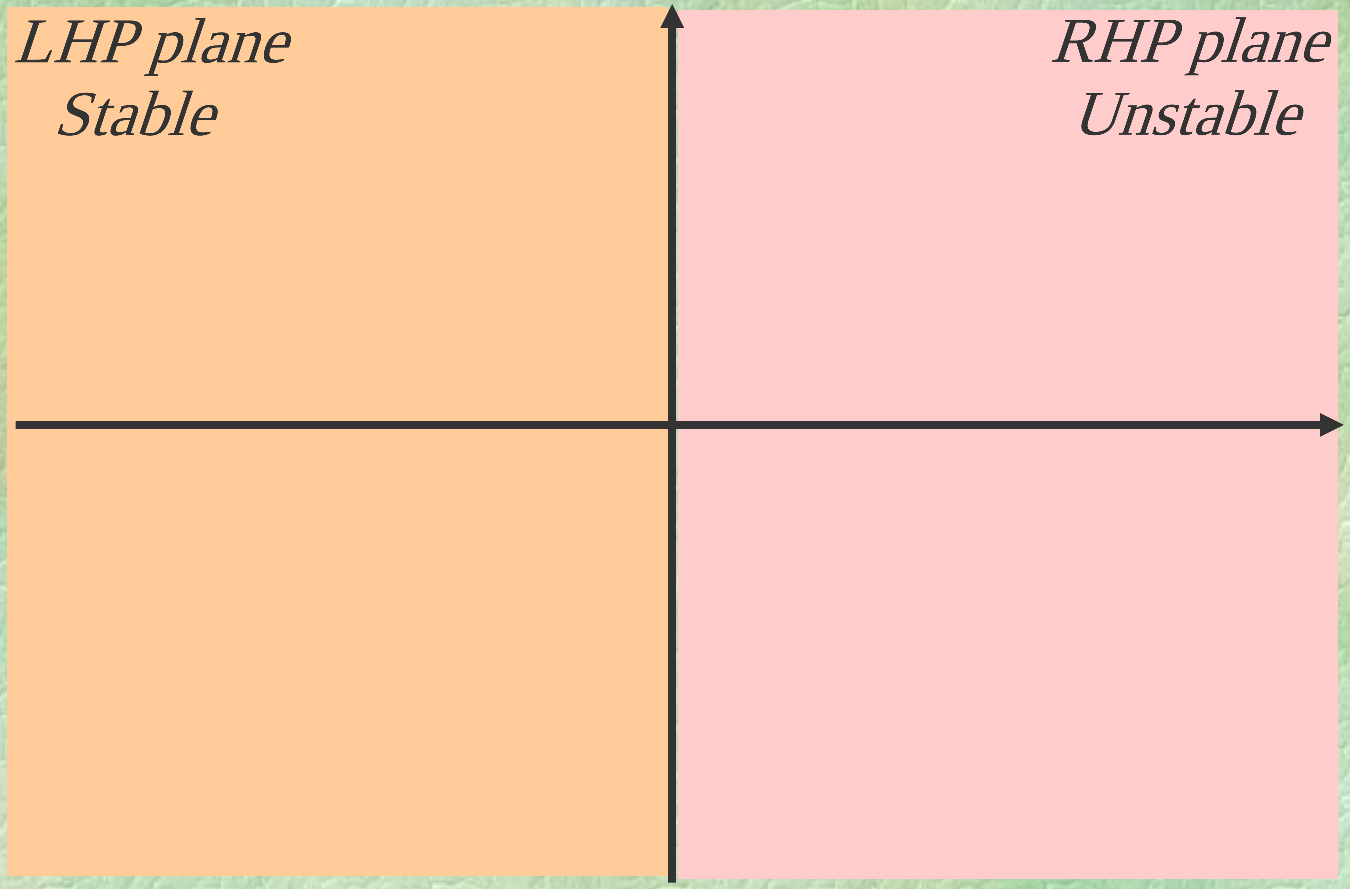
Let $|sI-A|=0$ → Find eigenvalues $\lambda_1, \lambda_2, ..., \lambda_n$

→ If $\lambda_1, \lambda_2, ..., \lambda_n \in$ LHP → System is asymptotically stable

For both kind of stability we need to compute the zero of some polynomial

Different regions in S plane

نواحی مختلف در صفحه S



Stability and Polynomial Analysis

پایداری و تجزیه تحلیل چند جمله ای ها

Consider a polynomial of the following form:

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The problem to be studied deals with the question of whether that polynomial has any root in RHP or on the jw axis.

مساله این است که آیا چند جمله ای فوق ریشه ای در RHP و یا روی محور jw دارد و یا خیر.

Some Polynomial Properties of Special Interest

چند خاصیت جالب چند جمله ای ها

Property 1: The coefficient a_{n-1} satisfies

$$a_{n-1} = -\sum_{i=1}^{n} \lambda_i$$

Property 2: The coefficient a_0 satisfies

$$a_0 = (-1)^n \prod_{i=1}^{n} \lambda_i$$

Property 3: If all roots of $p(s)$ have negative real parts, it is necessary that $a_i > 0$, $i \in \{0, 1, \ldots, n-1\}$.

Property 4: If any of the polynomial coefficients is nonpositive (negative or zero), then, one or more of the roots have nonnegative real plant.

Routh Hurwitz Algorithm

آلگوریتم روت هرویتز

$$p(s) = s^n + a_{n-1}s^{n-1} + \ldots + a_1 s + a_0$$

The Routh Hurwitz algorithm is based on the following numerical table.

s^n	1	a_{n-2}	a_{n-4}	....
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	....
s^{n-2}	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	....
s^{n-3}	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	....
.				
.				
.				
s^0	$\gamma_{n,1}$			

Routh's table

Routh Hurwitz Algorithm

آلگوریتم روت هرویتز

s^n	1	a_{n-2}	a_{n-4}	....
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	....
s^{n-2}	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	....
s^{n-3}	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	....
.				
.				
.				
s^0	$\gamma_{n,1}$			

Routh's table

$$\gamma_{2,1}=\frac{a_{n-2}a_{n-1}-a_{n-3}1}{a_{n-1}} \qquad \gamma_{2,2}=\frac{a_{n-4}a_{n-1}-a_{n-5}1}{a_{n-1}} \qquad \gamma_{2,3}=\frac{a_{n-6}a_{n-1}-a_{n-7}1}{a_{n-1}}$$

Result نتیجه

Consider a polynomial $p(s)$ and its associated table. Then the number of roots in RHP is equal to the number of sign changes in the first column of the table.

چند جمله ای $p(s)$ و جدول متناظر آن را در نظر بگیرید. تعداد ریشه های

واقع در RHP برابر با تعداد تغییر علامت در ستون اول جدول است.

Routh Hurwitz Algorithm

آلگوریتم روت هرویتز

s^n	1	a_{n-2}	a_{n-4}	....
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	....
s^{n-2}	$\gamma_{2,1}$	$\gamma_{2,2}$	$\gamma_{2,3}$	....
s^{n-3}	$\gamma_{3,1}$	$\gamma_{3,2}$	$\gamma_{3,3}$	....
.				
.				
.				
s^0	$\gamma_{n,1}$			

Routh's table

Number of sign changes=number of roots in RHP

Example 1: Check the number of zeros in the RHP

مثال ۱: تعداد صفر RHP سیستم زیر را تعیین کنید.

$$p(s) = 2s^4 + s^3 + 3s^2 + 5s + 10 = 0$$

s^4	2	3	10
s^3	1	5	0
s^2	-7	10	0
s^1	$\frac{45}{7}$	0	0
s^0	10	0	0

Two roots in RHP

Routh Hurwitz special cases

حالات خاص روت هرویتز

Routh Hurwitz special cases

1- The first element of a row is zero. (see example 2)

2- All elements of a row are zero. (see example 3)

Example 2: Check the number of zeros in the RHP

مثال ۲: تعداد صفر RHP سیستم زیر را تعیین کنید.

$$p(s) = s^4 + s^3 + 2s^2 + 2s + 3 = 0$$

s^4	1	2	3
s^3	1	2	0
s^2	~~0~~ ε	3	0
s^1	~~$\frac{2\varepsilon - 3}{\varepsilon}$~~ $\frac{-3}{\varepsilon}$	0	0
s^0	3	0	0

Two roots in RHP for any ε

Example 3: Check the number of zeros in the RHP

مثال ۳: تعداد صفر RHP سیستم زیر را تعیین کنید.

$$p(s) = s^5 + 4s^4 + 8s^3 + 8s^2 + 7s + 4 = 0$$

s^5	1	8	7
s^4	4	8	4
s^3	6	6	0
s^2	4	4	0
~~s^1~~	~~0~~	~~0~~	~~0~~
s^1	8	0	0
s^0	4	0	0

Auxiliary Polynomial

$$q(s) = 4s^2 + 4 = 0$$

$$q'(s) = 8s + 0$$

No RHP roots + two roots on imaginary axis

Remarks:

❖If all elements of the row s^{2n-1} are zero, there are 2n roots with same magnitude that they are symmetrical to the center of the s plane. It means they can be real roots with different signs or complex conjugate roots.

❖If there are no sign changes in first column, all roots of the auxiliary polynomial lie on the jω axis.

However, one important point to notice is that if there are repeated roots on the jω axis, the system is actually unstable.

$$p(s) = s^5 - 2s^4 - s + 2 = 0$$

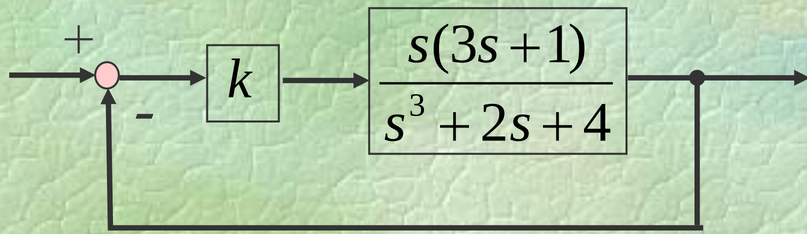
$$p(s) = s^5 - 2s^4 + 2s^3 - 4s^2 + s - 2 = 0$$

$$p(s) = s^5 + 2s^4 - s - 2 = 0$$

$$p(s) = s^5 + 2s^4 - 3s^3 - 6s^2 - 4s - 8 = 0$$

Example 4: Check the stability of following system for different values of *k*

مثال ۴: پایداری سیستم زیر را بر حسب مقادیر k بررسی کنید.



$$M(s)=\frac{k\dfrac{s(3s+1)}{s^3+2s+4}}{1+k\dfrac{s(3s+1)}{s^3+2s+4}}=\frac{ks(3s+1)}{s^3+2s+4+ks(3s+1)}$$

To check the stability we must check the RHP roots of

$$s^3+3ks^2+(2+k)s+4=0$$

s^3	1	$2+k$
s^2	$3k$	4
s^1	$\dfrac{3k(2+k)-4}{3k}$	0
s^0	4	0

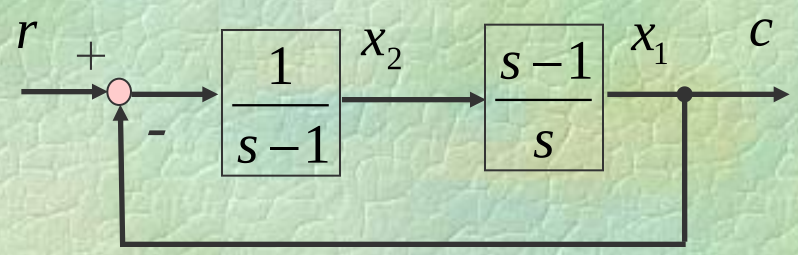
$$3k>0$$

$$\frac{3k^2+6k-4}{3k}>0$$

We need k>0.528 for stability

Example 5: Check the BIBO and internal stability of the following system.

مثال ۵ پایداری ورودی خروجی و پایداری داخلی سیستم زیر را تحقیق کنید.



BIBO stability

$$\frac{c(s)}{r(s)}=\frac{\frac{1}{s}}{1+\frac{1}{s}}=\frac{1}{s+1} \qquad p=-1 \qquad \Rightarrow$$

We have BIBO stability

Internal stability

$$\dot{x}_2-x_2=r-x_1 \qquad \dot{x}_1=-x_1+r$$

$$\dot{x}_1=\dot{x}_2-x_2 \qquad \dot{x}_2=-x_1+x_2+r$$

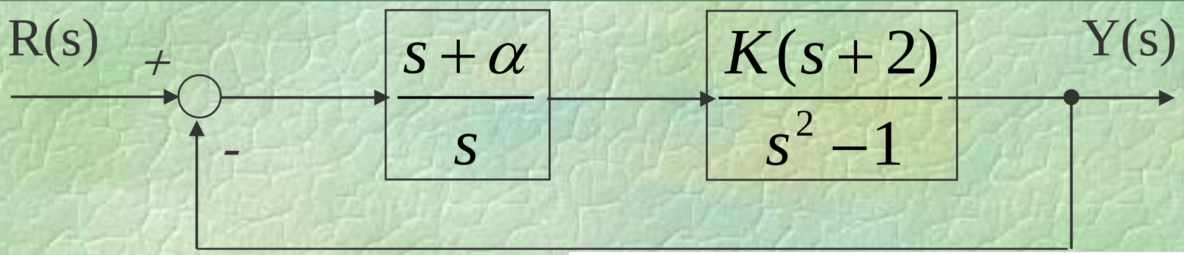
$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}-1 & 0\\ -1 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}r$$

$$|sI-A|=\begin{vmatrix}s+1 & 0\\ 1 & s-1\end{vmatrix}=s^2-1=0 \qquad \Rightarrow \qquad \lambda_1=1 \qquad \lambda_2=-1$$

We have not Internal stability

Example 6: The block Diagram of a control system is depicted in the following figure. Find the region in K-α plane concluding the system stable.

مثال ۶ :بلوک دیاگرام یک سیستم کنترل در شکل زیرنشان داده شده است. ناحیه ای در صفحه K-α به دست آورید که سیستم پایدار باشد.



The closed-loop transfer function is

$$\frac{Y(s)}{R(s)} = \frac{K(s+2)(s+\alpha)}{s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K}$$

The characteristic equation:

$$s^3 + Ks^2 + (2K + \alpha K - 1)s + 2\alpha K = 0$$

Routh Tabulation:

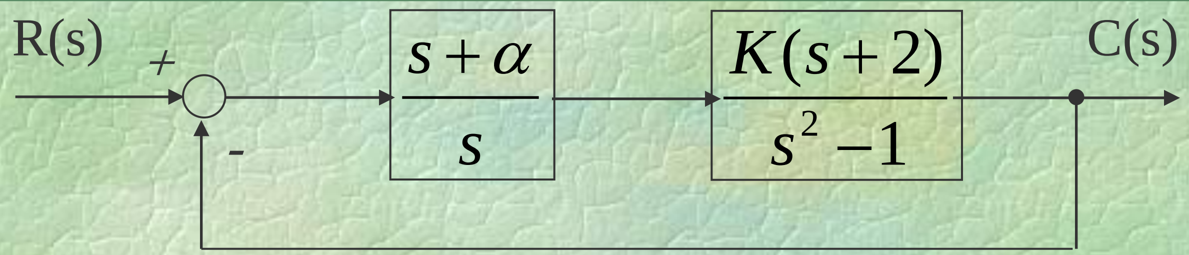
s^3	1	$2K + \alpha K - 1$
s^2	K	$2\alpha K$
s^1	$\frac{(2+\alpha)K^2 - K - 2\alpha K}{K}$	
s^0	$2\alpha K$	

$K > 0$

$(2+\alpha)K - 1 - 2\alpha > 0$

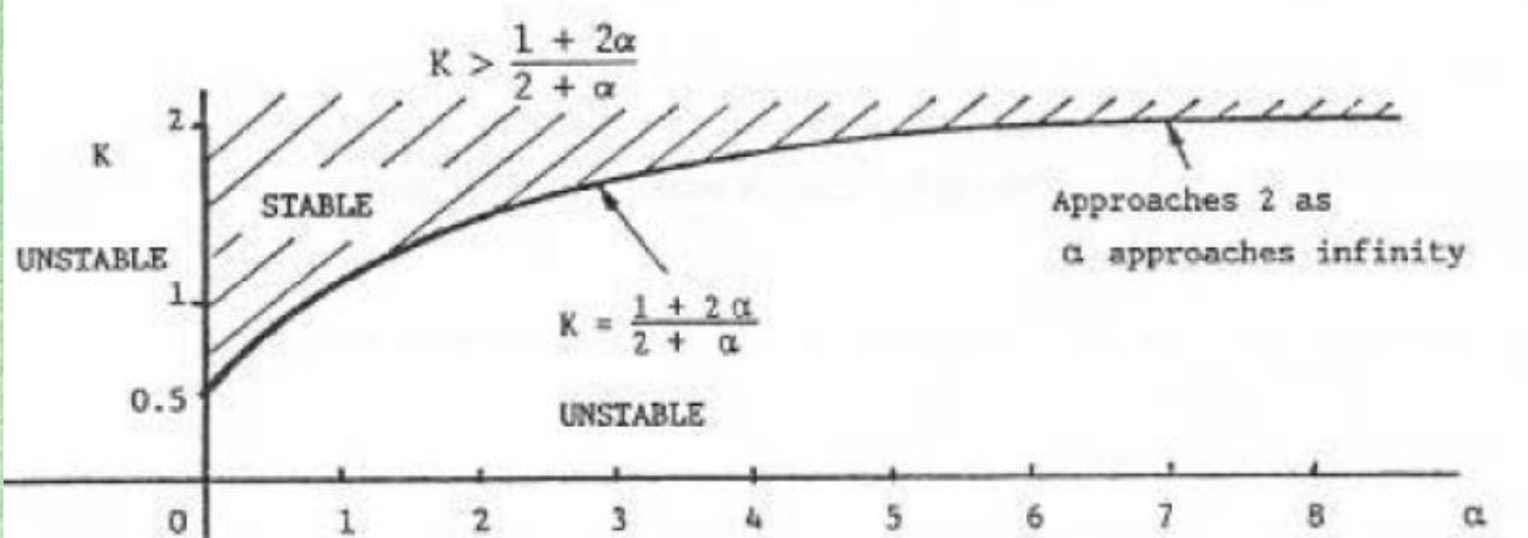
$\alpha > 0$

Example 6: The block Diagram of a control system is depicted in the following figure. Find the region in K-α plane concluding the system stable.



Stability Requirements: $\alpha > 0, \quad K > 0, \quad K > \dfrac{1+2\alpha}{2+\alpha}.$

K-versus-α Parameter Plane:



Exercises

1- Check the internal stability of following system.

$$\dot{x} = \begin{bmatrix} -2 & 1 & 3 \\ 0 & -3 & 2 \\ 1 & 4 & -1 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u$$
$$y = [1 \quad 2 \quad 0] x$$

2- a) Check the internal stability of following system.
b) Check the BIBO stability of following system.

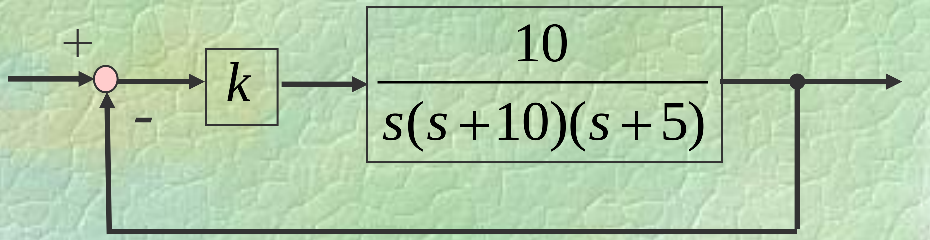
$$\dot{x} = \begin{bmatrix} -3 & -6 & -4 \\ 1 & 2 & 2 \\ -1 & -6 & -6 \end{bmatrix} x + \begin{bmatrix} 6 \\ -3 \\ 4 \end{bmatrix} u$$
$$y = [2 \quad 2 \quad -1] x$$

3- Are the real parts of all roots of following system less than -1.

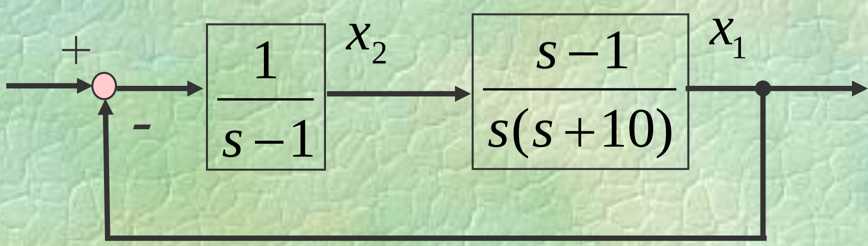
$$p(s) = s^5 + 4s^4 + 5s^3 + 6s^2 + 2s + 4$$

Exercises (Cont.)

4- Check the internal stability of following system versus *k*.



5- a)Check the BIBO stability of following system.
b) Check the internal stability of following system.



6- The eigenvalues of a system are -3,4,-5 and the poles of its transfer function are -3 and -5.(Midterm spring 2008)
a) Check the BIBO stability of following system.
b) Check the internal stability of following system.

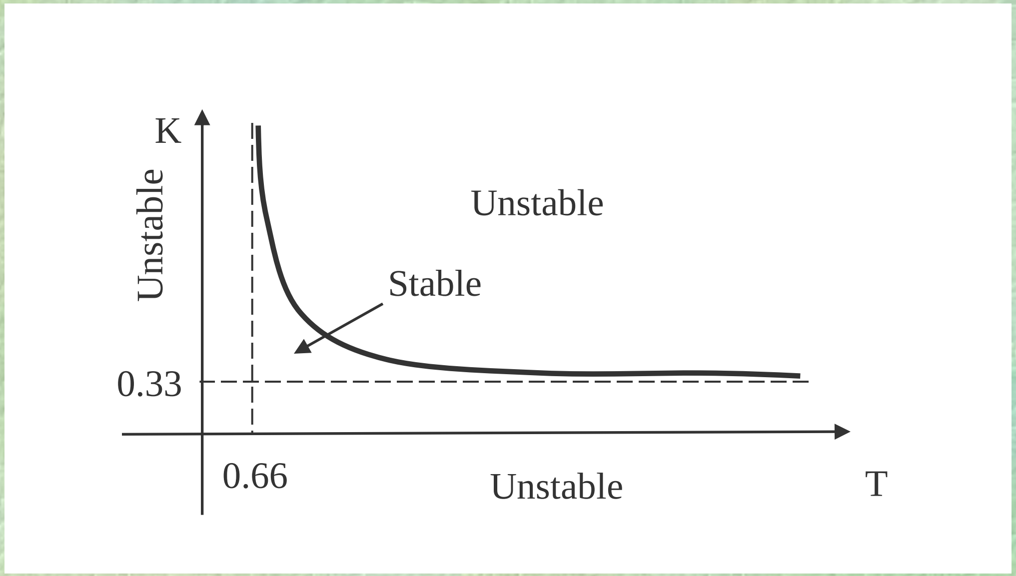
Exercises (Cont.)

7- The open-loop transfer function of a control system with negative unit feedback is:

$$G(s)H(s)=\frac{K(s+2)}{s(1+Ts)(1+2s)}$$

Find the region in K-T plane concluding the system stable.

Answer:



Exercises (Cont.)

8- Find the number of roots in the region [-2,2].

$$s^3 + 5s^2 + 11s + 15 = 0$$

9- The closed loop transfer function of a system is :

$$G(s) = \frac{k(s+2)}{s^4 + 7s^3 + 15s^2 + (k+25)s + 2k}$$

For the stability of the system which one is true? (k>0)

1) $0 \le k \le 28.12$

2) $0 < k < 28.12$

3) $0 \le k < 28.12$

4) $0 < k \le 28.12$

Answer: (4)