Lectures on Multivariable Feedback Control

Ali Karimpour

Department of Electrical Engineering, Faculty of Engineering, Ferdowsi University of Mashhad (September 2009)

Chapter 3: Limitation on Performance in MIMO Systems

In this chapter, we discuss the fundamental limitations on performance in SISO systems. Proper scaling of the input, output and disturbance variables prior to this analysis is critical. Consider the simple one degree-of-freedom configuration in Figure 3-1. The output of the system is

$$y = \underbrace{(I+G(s)K(s))^{-1}G(s)K(s)}_{T}r + \underbrace{(I+G(s)K(s))^{-1}}_{S}G_d(s)d - \underbrace{(I+G(s)K(s))^{-1}G(s)K(s)}_{T}n \qquad \text{3-1}$$

For "perfect control" we want $e = y - r = 0$ that is, we would like

$$S = 0,\quad T = I \qquad \text{3-2}$$

For disturbance rejection and command tracking we need $S \approx 0$ or equivalently $T \approx I$. On the other hand, the requirement for zero noise transmission implies that $T \approx 0$ or equivalently $S \approx I$.

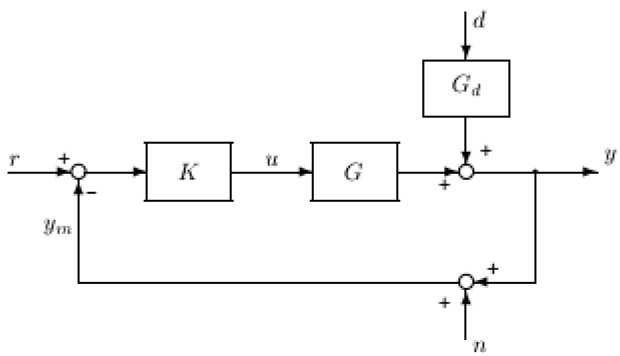

Figure 3-1 One degree-of-freedom configuration

This illustrates the fundamental nature of feedback design which always involves a trade-off between conflicting objectives, in this case between large loop gains for disturbance rejection and tracking, and small loop gains to reduce the effect of noise.

Some important design objectives which necessitate trade-offs in feedback control are:

1- Performance, good disturbance rejection: needs $T \approx I$ or $S \approx 0$.

2- Performance, good command following: needs $T \approx I$ or $S \approx 0$.

3- Mitigation of measurement noise on plant outputs: needs $T \approx 0$ or $S \approx I$.

Fortunately, the conflicting design objectives mentioned above are generally in different frequency ranges.

3-1 Scaling and Performance

Throughout this chapter and the next, when we will assume that the variables and models have been scaled as outlined previously so that the requirement for acceptable performance is:

- For any reference $r(t)$ between -*R* and *R* and any disturbance $d(t)$ between -1 and 1, keep the output $y(t)$ within the range $r(t)-1$ to $r(t)+1$ (at least most of the time), using an input $u(t)$ within the range -1 to 1.

We will interpret this definition from a frequency-by-frequency sinusoidal point of view, i.e. $d(t) = \sin \omega t$ and so on. With $e = y - r$ we then have:

- For any disturbance $|d(\omega)| \le 1$ and any reference $|r(\omega)| \le R(\omega)$, the **performance requirement** is to keep at each frequency ω the control error $|e(\omega)| \le 1$ using an input $|u(\omega)| \le 1$.

It is impossible to track very fast reference changes, so we will assume that $R(\omega)$ is frequency dependent, for simplicity, we assume that $R(\omega)$ is *R* (a constant) up to the frequency ω_r and is 0 above that frequency.

It could also be argued that the magnitude of the sinusoidal disturbances should approach zero at high frequencies. While this may be true, we really only care about frequencies within the bandwidth of the system, and in most cases it is reasonable to assume that the plant experiences sinusoidal disturbances of constant magnitude up to this frequency. Similarly, it might also be argued that the allowed control error should be frequency dependent. For example, we may require no steady-state offset, i.e. *e* should be zero at low frequencies. However, including frequency variations is not recommended when doing a preliminary analysis (however, one may take such considerations into account when interpreting the results).

Recall that with $r = R\tilde{r}$ the control error may be written as

$$e = y - r = Gu + G_d d - R\tilde{r} \qquad \text{3-3}$$

where *R* is the magnitude of the reference and $|\tilde{r}(\omega)| \le 1$ and $|d(\omega)| \le 1$ are unknown signals. We will use 3-3 to unify our treatment of disturbances and references. Specifically, we will derive results for disturbances, which can then be applied directly to the references by replacing G_d by $-R$.

3-2 Shaping Closed-loop Transfer Functions

In this section, we introduce the reader to shaping of closed-loop transfer functions where we synthesize a controller by minimizing an H_∞ performance objective. Many design procedure act on the shaping of the open-loop transfer function *L*. An alternative design strategy is to directly shape the magnitudes of closed-loop transfer functions, such as $S(s)$ and $T(s)$. Such a design strategy can be formulated as an H_∞ optimal control problem, thus automating the actual controller design and leaving the engineer with the task of selecting reasonable bounds "weights" on the desired closed-loop transfer functions. Before explaining how this may be done in practice, we discuss the terms H_∞.

3-2-1 The terms H_∞ and H_2

The H_∞ norm of a stable scalar transfer function matrix $F(s)$ is simply define as,

$$\|F(s)\|_\infty \cong \max_\omega \bar{\sigma}\big(F(j\omega)\big) \tag{3-4}$$

Strictly speaking, we should here replace "max" (the maximum value) by "sup" the supremum (the least upper bound). This is because the maximum may only be approached as $\omega \to \infty$ and may therefore not actually be achieved. However, for engineering purposes there is no difference between "sup" and "max".

The terms H_∞ norm and H_∞ control are intimidating at first, and a name conveying the engineering significance of H_∞ would have been better. After all, we are simply talking about a design method which aims to press down the peak(s) of one or more selected transfer functions. However, the term H_∞ which is purely mathematical, has now established itself in the control community. In literature the symbol H_∞ stands for the transfer function matrices with bounded ∞-norm which is the set of stable and proper transfer function matrices.

Similarly, the symbol H_2 stands for the transfer function matrices with bounded 2-norm, which is the set of stable and strictly proper transfer function matrices. The H_2 norm of a strictly proper stable transfer function matrix is defined as

$$\|F(s)\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\infty}^{+\infty} tr\left[F(j\omega)F(j\omega)^H\right]d\omega} \qquad 3\text{-}5$$

Note that the H_2 norm of a semi-proper transfer function (where $\lim_{\omega\to\infty} F(j\omega)$ is a nonzero constant matrix) is infinite, whereas its H_∞ norm is finite.

3-2-2 Weighted Sensitivity

As already discussed, the sensitivity function S is a very good indicator of closed-loop performance (both for SISO and MIMO systems). The main advantage of considering S is that because we ideally want S to be small, it is sufficient to consider just its magnitude, $\|S\|$ that is, we need not worry about its phase. Typical specifications in terms of S include:

1- Minimum bandwidth frequency ω_B^* (defined as the frequency where $\bar{\sigma}(S)$ crosses 0.707 from below.
2- Maximum tracking error at selected frequencies.
3- System type, or alternatively the maximum steady-state tracking error, *A*.
4- Shape of *S* over selected frequency ranges.
5- Maximum peak magnitude of S, $\|S(j\omega)\|_\infty \le M$.

The peak specification prevents amplification of noise at high frequencies, and also introduces a margin of robustness; typically we select $M = 2$. Mathematically, these specifications may be captured simply by an upper bound, $\frac{1}{|w_p(s)|}$, on the magnitude of S where $w_p(s)$ is a weight selected by the designer. The subscript *P* stands for performance since *S* is mainly used as a performance indicator, and the performance requirement becomes

$$\bar{\sigma}\big(S(j\omega)\big) \le \frac{1}{|w_P(j\omega)|}, \qquad \forall\omega \tag{3-6}$$

$$\Leftrightarrow \qquad \bar{\sigma}\big(w_P(j\omega)S(j\omega)\big) \le 1, \quad \forall\omega \qquad \Leftrightarrow \qquad \|w_P(j\omega)S(j\omega)\|_\infty \le 1 \tag{3-7}$$

The last equivalence follows from the definition of the H_∞ norm, and in words the performance requirement is that the H_∞ norm of the weighted sensitivity, $w_P S$, must be less than one. In Figure 3-2(a) an example is shown where the sensitivity, $|S|$ exceeds its upper bound $\frac{1}{|w_P|}$ at some frequencies. The resulting weighted sensitivity $|w_P S|$ therefore exceeds 1 at the same frequencies as is illustrated in Figure 3-2(b). Note that we usually do not use a log-scale for the magnitude when plotting weighted transfer functions, such as $|w_P S|$.

Weight selection: An asymptotic plot of a typical upper bound $\frac{1}{|w_P|}$ is shown in Figure 3-3. The weight illustrated may be represented by

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A} \tag{3-8}$$

and we see that $|w_P(j\omega)|^{-1}$ is equal to $A \le 1$ at low frequencies, is equal to $M \ge 1$ at high frequencies, and the asymptote crosses 1 at the frequency, ω_B^*, which is approximately the bandwidth requirement. For this weight the loop shape $L = \frac{\omega_B^*}{s}$ yields an S which exactly matches the bound 3-7 at frequencies below the bandwidth and easily satisfies (by a factor M) the bound at higher frequencies. This L has a slope -1 in the frequency range below crossover. In some cases, in order to improve performance, we may want a steeper slope for L (and S) below

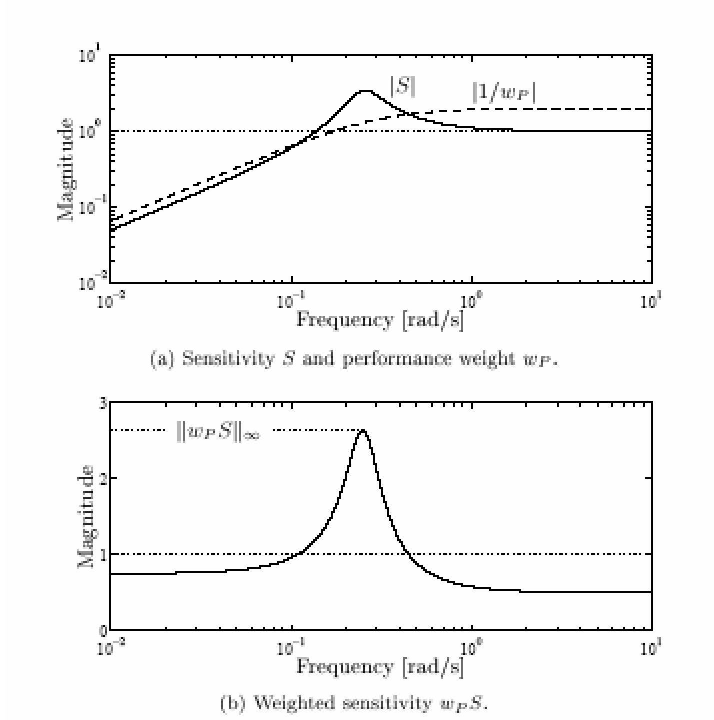

Figure 3-2 Case where $|S|$ exceeds its bound $\frac{1}{|w_P|}$, resulting in $\|w_P S\|_\infty > 1$

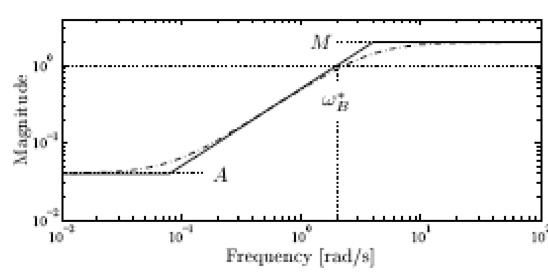

Figure 3-3 Inverse of performance weight. Exact and asymptotic plot of $\frac{1}{|w_P(j\omega)|}$ in 3-8

the bandwidth, and then a higher-order weight may be selected. A weight which asks for a slope -2 for L at lower frequencies is

$$w_P(s) = \frac{\left(s/M^{1/2} + \omega_B^*\right)^2}{\left(s + \omega_B^* A^{1/2}\right)^2} \qquad \text{3-9}$$

The insight gained from the previous section on loop-shaping design is very useful for selecting weights. For example, for disturbance rejection we must satisfy $\bar{\sigma}\left(SG_d(j\omega)\right) < 1$ at all frequencies (assuming the variables have been scaled to be less than 1 in magnitude). It then follows that a good initial choice for the performance weight is to let $w_p(s)$ look like $|G_d(j\omega)|$ at frequencies where $|G_d| > 1$. In other cases, one may first obtain an initial controller using another design procedure, and the resulting sensitivity $\bar{\sigma}(S(j\omega))$ may then be used to select a performance weight for a subsequent H_∞ design.

3-2-3 Stacked Requirements: Mixed Sensitivity

The specification $\|\omega_P S\|_\infty < 1$ puts a lower bound on the bandwidth, but not an upper one, and nor does it allow us to specify the roll-off of $L(s)$ above the bandwidth. To do this one can make demands on another closed-loop transfer function, for example, on the complementary sensitivity $T = I - S = GKS$.

Also, to achieve robustness or to avoid too large input signals, one may want to place bounds on the transfer function KS.

For instance, one might specify an upper bound $\frac{1}{|\omega_T|}$ on the magnitude of T to make sure that L rolls off sufficiently fast at high frequencies, and an upper bound, $\frac{1}{|\omega_u|}$ on the magnitude of KS to restrict the size of the input signals, $u = KS(r - G_d d)$. To combine these "mixed sensitivity" specifications, a "stacking approach" is usually used, resulting in the following overall specification:

$$\|N\|_\infty = \max_\omega \bar{\sigma}(N(j\omega)) < 1, \quad N = \begin{bmatrix} w_P S \\ w_T T \\ w_u KS \end{bmatrix} \qquad 3\text{-}10$$

We here use the maximum singular value $\bar{\sigma}(N(j\omega))$ to measure the size of the matrix N at each frequency. For SISO systems, N is a vector and $\bar{\sigma}(N)$ is the usual Euclidean vector norm:

$$\bar{\sigma}(N) = \sqrt{|w_P S|^2 + |w_T T|^2 + |w_u KS|^2} \qquad 3\text{-}11$$

The stacking procedure is selected for mathematical convenience as it does not allow us to exactly specify the bounds on the individual transfer functions as described above. For example, assume that $\phi_1(K)$ and $\phi_2(K)$ are two functions of K (which might represent $\phi_1(K) = w_P S$ and $\phi_2(K) = w_T T$) and that we want to achieve

$$|\phi_1| < 1 \text{ and } |\phi_2| < 1 \qquad \text{3-12}$$

This is similar to, but not quite the same as the stacked requirement

$$\bar{\sigma}\begin{bmatrix}\phi_1 \\ \phi_2\end{bmatrix} = \sqrt{|\phi_1|^2 + |\phi_2|^2} < 1 \qquad \text{3-13}$$

Objectives 3-12 and 3-13 are very similar when either $|\phi_1|$ or $|\phi_2|$ is small but in the worst case when $|\phi_1| = |\phi_2|$ we get from 3-13 that $|\phi_1| < 0.707$ and $|\phi_2| = 0.707$. That is, there is a possible "error" in each specification equal to at most a factor $\sqrt{2} = 3dB$. In general, with n stacked requirements the resulting error is at most $\sqrt{n}$. This inaccuracy in the specifications is something we are probably willing to sacrifice in the interests of mathematical convenience. In any case, the specifications are in general rather rough, and are effectively knobs for the engineer to select and adjust until a satisfactory design is reached.

After selecting the form of N and the weights, the H_∞ optimal controller is obtained by solving the problem

$$\min_K \|N(K)\|_\infty \qquad \text{3-14}$$

where K is a stabilizing controller. Let $\gamma_0 = \min_K \|N(K)\|_\infty$ denote the optimal H_∞ norm. An important property of H_∞ optimal controllers is that they yield a flat frequency response, that is, $\bar{\sigma}(N(j\omega)) = \gamma_0$ at all frequencies. The practical implication is that, except for at most a factor $\sqrt{n}$ the transfer functions resulting from a solution to 3-14 will be close to γ_0 times the bounds selected by the designer. This gives the designer a mechanism for directly shaping the magnitudes of $\bar{\sigma}(S)$, $\bar{\sigma}(T)$, $\bar{\sigma}(KS)$, and so on.

Example 3-1: H_∞ mixed sensitivity design for the disturbance process.

Consider the disturbance process shown in the Figure 3-4. The system is described by

$$G(s) = \frac{200}{10s+1}\frac{1}{(0.05s+1)^2}\ , \qquad G_d(s) = \frac{100}{10s+1}$$

The control objectives are:

1- Command tracking: The rise time (to reach 90% of the final value) should be less than 0.3 second and the overshoot should be less than 5%.

2- Disturbance rejection: The output in response to a unit step disturbance should remain within the range $[-1,1]$ at all times, and it should return to 0 as quickly as possible ($|y(t)|$ should at least be less than 0.1 after 3 seconds).

3- Input constraints: *u(t)* should remain within the range $[-1,1]$ at all times to avoid input saturation (this is easily satisfied for most designs).

Since $G_d(0) = 100$ we have that without control the output response to a unit disturbance (d=1) will be 100 times larger than what is deemed to be acceptable. The magnitude $|G_d(j\omega)|$ is lower at higher frequencies, but it remains larger than 1 up to $\omega_d = 10[rad/\sec]$ (where $|G_d(j\omega_d)| = 1$).

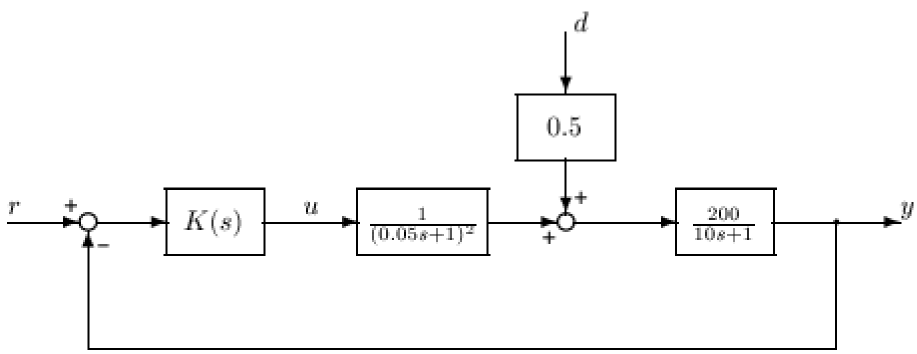

Figure 3-4 Block diagram representing the disturbance process of example 3-1

Thus, feedback control is needed up to frequency ω_d so we need ω_c to be approximately equal to 10 rad/sec for disturbance rejection. On the other hand, we do not want ω_c to be larger than necessary because of sensitivity to noise and stability problems associated with high gain feedback. We will thus aim at a design with $\omega_c \approx 10\, rad/\sec$.

Consider an H_∞ mixed sensitivity *S/KS* design in which

$$N = \begin{bmatrix} w_P S \\ w_u KS \end{bmatrix} \qquad 3\text{-}15$$

It was stated earlier that appropriate scaling has been performed so that the inputs should be about 1 or less in magnitude, and we therefore select a simple input weight $\omega_u = 1$. The performance weight is chosen, in the form of 3-8 as

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}, \quad M = 1.5, \quad \omega_B^* = 10, \quad A = 10^{-4} \qquad 3\text{-}16$$

A value of A=0 would ask for integral action in the controller, but to get a stable weight and to prevent numerical problems in the algorithm used to synthesize the controller, we have moved the integrator slightly by using a small non-zero value for A. This has no practical significance in terms of control performance. The value $\omega_B^* = 10$ has been selected to achieve approximately the desired crossover frequency $\omega_c \approx 10\, rad/\sec$. The H_∞ problem is solved with the $\mu-$toolbox in MATLAB using the commands in table 3-1.

Table 3-1 MATLAB program to synthesize an H_∞ controller.

```
% Uses the Mu_toolbox
G=nd2sys(1,conv([10 1],conv([0.05 1],[0.05 1])),200);                %Plant is G.
M=1.5; wb=10; A=1.e-4;Wp = nd2sys([1/M wb],[1 wb*A]); Wu=1;      % Weights.
%
%  Generalized plant P is found with function sysic;
%
Systemnames='G Wp Wu';
inputvar='[r(1);u(1)]';
outputvar='[Wp;Wu;r-G]';
input_to_G='[u]';
input_ to_ Wp='[r-G]';
input_to_Wu='[u]';
sysoutname='P';
```

```
cleanupsysic='yes';
sysic;
%
%  Find H-infinity optimal controller;
%
Nmeas=1; nu=1; gmn=0.5;gmx=20; tol=0.001;
[khinf,ghinf,gopt]= hinfsyn(P,nmeas,nu,gmn,gmx,tol);
```

For this problem, we achieved an optimal H_∞ norm of 1.37, so the weighted sensitivity requirements are not quite satisfied (see design 1 in Figure 3-5). Nevertheless, the design seems good with $M_S = 1.30$, $M_T = 1.0$, $GM = 8.04$, $PM = 71.2^\circ$ and $\omega_c = 7.22$, and the tracking response is very good as shown by curve y_1 in Figure 3-6 (a). However, we see from curve y_1 in Figure 3-6 (b) that the disturbance response is very sluggish. If disturbance rejection is the main concern, then from our earlier discussion we need for a performance weight that specifies higher gains at low frequencies. We therefore try

$$w_P(s) = \frac{\left(s/M^{1/2} + \omega_B^*\right)^2}{\left(s + \omega_B^* A^{1/2}\right)^2}, \quad M = 1.5, \; \omega_B^* = 10, \; A = 10^{-6} \qquad \text{3-17}$$

The inverse of this weight is shown in Figure 3-5 and is seen from the dashed line to cross 1 in magnitude at about the same frequency as weight w_{P1}, but it specifies tighter control at lower frequencies. With the weight w_{P2} we get a design with an optimal H_∞ norm of 2.21, yielding $M_S = 1.63$, $M_T = 1.43$, $GM = 4.76$, $PM = 43.3^\circ$ and $\omega_c = 11.34$. The disturbance response is very good, whereas the tracking response has a somewhat high overshoot, see curve y_2 in Figure 3-6(a). In conclusion, design 1 is best for reference tracking whereas design 2 is best for disturbance rejection. Two get a design with both good tracking and good disturbance rejection we need a two degrees-of-freedom controller.

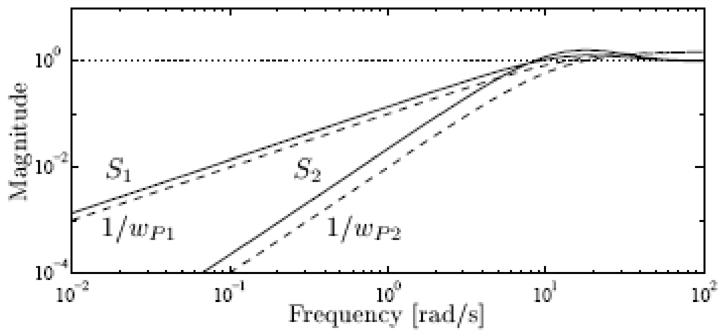

Figure 3-5 Inverse of performance weight (dashed line) and resulting sensitivity function (solid line) for two H_∞ design (1 and 2) for the disturbance process

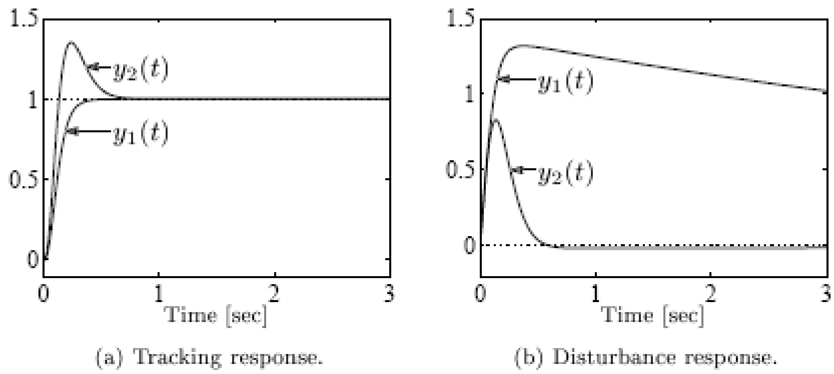

Figure 3-6 Closed-loop step responses for two alternative H_∞ designs (1 and 2) for the disturbance process

3-3 Fundamental Limitation on Sensitivity

3-3-1 *S* plus *T* is the identity matrix

From the identity $S + T = I$ we get

$$|\bar{\sigma}(S) - 1| \le \bar{\sigma}(T) \le \bar{\sigma}(S) + 1 \qquad \text{3-18}$$

$$|\bar{\sigma}(T) - 1| \le \bar{\sigma}(S) \le \bar{\sigma}(T) + 1 \qquad \text{3-19}$$

This shows that we cannot have both *S* and *T* small (close to 0) simultaneously. The magnitude of $\bar{\sigma}(T)$ and $\bar{\sigma}(S)$ differs only at most 1 at a given frequency, so $\bar{\sigma}(T)$ is large if and only if $\bar{\sigma}(S)$

is large. For example, if $\bar{\sigma}(T)$ is 5 at a given frequency, then $\bar{\sigma}(S)$ must be between 4 and 6 at this frequency.

3-3-2 Interpolation Constraints

RHP-zero: If *G(s)* has a RHP-zero at *z* with output direction y_z then for internal stability of the feedback system the following interpolation constraints must apply:

$$y_z^H T(z) = 0; \quad y_z^H S(z) = y_z^H \qquad 3\text{-}20$$

In words, it says that *T* must have a RHP-zero in the same direction as *G* and that *S(z)* has an eigenvalue of 1 corresponding to the left eigenvector y_z.

Proof of 3-20: Since *z* is a RHP-zero of *G(s)* with output direction y_z then we have $y_z^H G(z) = 0$. For internal stability, the controller cannot cancel the RHP-zero and it follows that $L = GK$ has a RHP-zero in the same direction, i.e. $y_z^H L(z) = 0$. Now $S = (I + L)^{-1}$ is stable and thus has no RHP-pole at $s = z$. It then follows from $T = LS$ that $y_z^H T(z) = 0$ and $y_z^H (I - S(z)) = 0$

RHP-pole: If *G(s)* has a RHP pole at *p* with output direction y_p then for internal stability the following interpolation constraints apply

$$S(p) y_p = 0; \quad T(p) y_p = y_p \qquad 3\text{-}21$$

Proof of 3-21: The square matrix $L(p)$ has a RHP-pole at $s = p$ and if by definition of RHP-pole there exists an output pole direction y_p such that $L^{-1}(p) y_p = 0$. Since *T* is stable, it has no RHP-pole at $s = p$ so $T(p)$ is finite. It then follows, from $S = TL^{-1}$, that $S(p) y_p = T(p) L^{-1}(p) y_p = 0$ and $T(p) y_p = (I - S(p)) y_p = y_p$.

3-3-3 Sensitivity Integrals

For SISO systems we have several integral constraints on the sensitivity (the waterbed effects). These may be generalized to MIMO systems by using the determinant or singular value of *S*. For example, the generalization of the Bode sensitivity integral may be written

$$\int_0^\infty \ln\left|\det S(j\omega)\right| d\omega = \sum_i \int_0^\infty \ln \sigma_i \left(S(j\omega)\right) d\omega = \pi . \sum_{i=1}^{N_p} \mathrm{Re}(p_i) \qquad 3\text{-}22$$

where p_i is RHP-pole of G(s) and N_p is the number of RHP-pole of G(s). For a stable L(s) the integral is zero. The area of $\ln\sigma_i(S(j\omega))$ is negative, must equal the area of $\ln\sigma_i(S(j\omega))$ is positive for stable system. So the area of $\sigma_i(S(j\omega))$ is less than one, must equal the area of $\sigma_i(S(j\omega))$ above one for stable system. In this respect, the benefits and costs of feedback are balanced exactly, as in the waterbed analogy. From this we expect that an increase in the bandwidth ($\sigma_i(S(j\omega))$ smaller than 1 over a larger frequency range) must come at the expense of a larger peak in $\sigma_i(S(j\omega))$. The presence of unstable poles usually increases the peak of the sensitivity $\sigma_i(S(j\omega))$ as seen from the positive contribution of $\pi.\sum_{i=1}^{N_p}\mathrm{Re}(p_i)$ in 3-22.

Specifically, the area of sensitivity increase ($\sigma_i(S(j\omega))$>1) exceeds that of sensitivity reduction by an amount proportional to the sum of the distance from the unstable poles to the left-half plane. This is plausible since we might expect to have to pay a price for stabilizing the system.

3-4 Fundamental Limitation: Bounds on Peaks

Based on the above interpolation constraints we here derive lower bounds on various closed-loop transfer function matrices.

In the following, $M_{S,\min}$ and $M_{T,\min}$ denote the lowest achievable values for $\|S\|_\infty$ and $\|T\|_\infty$, respectively, using any stabilizing controller *K*. That is, we define

$$M_{S,\min} \cong \min\|S\|_\infty,\ M_{T,\min} \cong \min\|T\|_\infty$$

Theorem 3-1 Sensitivity and Complementary Sensitivity Peaks.

Consider a rational plant *G(s)* (with no time delay). Let z_i be the one of the N_z RHP-zeros of *G(s)* with output zero direction vectors $y_{z,i}$. Let p_i be the one of the N_p RHP-poles of *G(s)* with output pole direction vectors $y_{p,i}$. Furthermore, assume that z_i and p_i are all distinct. Then we have the following tight lower bound on $\|S\|_\infty$ and $\|T\|_\infty$:

$$M_{S,\min} = M_{T,\min} = \sqrt{1+\bar{\sigma}^2\left(Q_z^{-1/2}Q_{zp}Q_p^{-1/2}\right)} \qquad 3\text{-}23$$

where the Q_z, Q_p and Q_{zp} are $N_z \times N_z$, $N_p \times N_p$ and $N_z \times N_p$ matrices respectively, and there elements are:

$$[Q_z]_{ij} = \frac{y_{z,i}^H y_{z,j}}{z_i + \bar{z}_j},\ [Q_p]_{ij} = \frac{y_{p,i}^H y_{p,j}}{\bar{p}_i + p_j},\ [Q_{zp}]_{ij} = \frac{y_{z,i}^H y_{p,j}}{z_i - p_j} \qquad \text{3-24}$$

Note that 3-23 gives a tight bound for any number of RHP-poles and RHP-zeros.

Example 3-2

Consider the SISO plant

$$G(s) = \frac{(s-1)(s-3)}{(s-2)(s+1)^2}$$

Derive lower bounds on $\|S\|_\infty$ and $\|T\|_\infty$.

Solution:

For this plant we have $z_1 = 1,\ z_2 = 3,\ p_1 = 2$, and since this is a SISO plant, all direction vectors are 1. Since we have RHP-zeros close to RHP-pole we except that control is fundamentally difficult. This is verified from theorem 3-1 that

$$Q_z = \begin{bmatrix} 1/2 & 1/4 \\ 1/4 & 1/6 \end{bmatrix},\ Q_p = 1/4,\ Q_{z,p} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

$$M_{S,\min} = M_{T,\min} = \sqrt{1 + \bar{\sigma}^2\left(\begin{bmatrix} -7.9531 \\ 12.6786 \end{bmatrix}\right)} = 15$$

We see from the factor $\dfrac{y_{z,i}^H y_{p,j}}{z_i - p_j}$ in Q_{zp}, that the bound will be large if we have a RHP-pole p_i close to RHP-zero z_j and with directions aligned such that $y_{z,i}^H y_{p,j}$ is not small.

Example 3-3

Consider the MIMO plant

$$G_{30}(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \begin{bmatrix} \cos(30^\circ) & -\sin(30^\circ) \\ \sin(30^\circ) & \cos(30^\circ) \end{bmatrix} \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+2}{0.1s+1} \end{bmatrix};\ z = 2,\ p = 3$$

Derive lower bounds on $\|S\|_\infty$ and $\|T\|_\infty$.

Solution:

The output direction vectors corresponding to the RHP-zero at $z = 2$ and the RHP-pole at $p = 2$ are, respectively,

$$y_z = \begin{bmatrix} 0.327 \\ -0.945 \end{bmatrix}, \quad y_p = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

This is verified from theorem 3-1 that

$$Q_z = 1/4,\ Q_p = 1/6,\ Q_{z,p} = 0.327$$

$$M_{S,\min} = M_{T,\min} = \sqrt{1 + \bar{\sigma}^2(1.6020)} = 1.8885$$

One RHP-pole and one RHP-zero: For a plant with one RHP-zero z and one RHP-pole *p*, 3-23 converts to:

$$M_{S,\min} = M_{T,\min} = \sqrt{\sin^2\phi + \frac{|z+p|^2}{|z-p|^2}\cos^2\phi} \qquad \text{3-25}$$

where $\phi = \cos^{-1}\left|y_z^H y_p\right|$ is the angle between the output directions of the pole and zero. If the pole and zero are orthogonal to each other $\phi = 90^\circ$, and $M_{S,\min} = M_{T,\min} = 1$, and there is no additional penalty for having both a RHP-pole and a RHP-zero.

Example 3-4

Consider the MIMO plant

$$G_\alpha(s) = \begin{bmatrix} \frac{1}{s-p} & 0 \\ 0 & \frac{1}{s+3} \end{bmatrix} \underbrace{\begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}}_{U_\alpha} \begin{bmatrix} \frac{s-z}{0.1s+1} & 0 \\ 0 & \frac{s+2}{0.1s+1} \end{bmatrix};\ z = 2,\ p = 3$$

For $\alpha = 0^\circ$ the rotation matrix $U_0 = I$, and the plant consists of two decoupled subsystems

$$G_0(s) = \begin{bmatrix} \frac{s-z}{(0.1s+1)(s-p)} & 0 \\ 0 & \frac{s+2}{(0.1s+1)(s+3)} \end{bmatrix}$$

Here the subsystem $g_{11}(s)$ has both a RHP-pole and a RHP-zero and closed-loop performance is expected to be poor. On the other hand, there are no particular control problems related to the subsystem $g_{22}(s)$. Next, consider $\alpha = 90^\circ$, for which we have

$$U_{90} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad and \quad G_{90}(s) = \begin{bmatrix} 0 & -\dfrac{s+2}{(0.1s+1)(s-p)} \\ \dfrac{s-z}{(0.1s+1)(s+3)} & 0 \end{bmatrix}$$

and we again have two decoupled subsystems, but this time in the off-diagonal elements. The main difference, however, is that there is no interaction between the RHP-pole and RHP-zero in this case, so we expect this plant to be easier to control. For intermediate values of α we do not have decoupled subsystems, and there will be some interaction between the RHP-pole and RHP-zero.

Since the RHP-pole in $G_\alpha(s)$ is located at the output of the plant, its output direction is fixed and we find $y_p = [1 \quad 0]^T$ for all values of α. On the other hand, the RHP-zero output direction changes from $[1 \quad 0]^T$ for $\alpha = 0^\circ$ to $[0 \quad 1]^T$ for $\alpha = 90^\circ$. Thus, the angle ϕ between the pole and zero direction also varies between 0° and 90° but ϕ and α are not equal. This is seen from the Table 3-2 where we also give $M_{S,\min} = M_{T,\min}$ for four rotation angles $\alpha = 0^\circ$, 30°, 60° and $\alpha = 90^\circ$.

The table also shows the values of $\|S\|_\infty$ and $\|T\|_\infty$ obtained by an H_∞ optimal *S/KS* design(see section 3-2-3) using the following weights

Table 3-2 Result of example 3-4

α	0°	30°	60°	90°
y_z	$\begin{bmatrix} 1 \\ 0 \end{bmatrix}$	$\begin{bmatrix} 0.33 \\ -0.94 \end{bmatrix}$	$\begin{bmatrix} 0.11 \\ -0.99 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$
$\phi = \cos^{-1}\left(y_z^H y_p\right)$	0°	70.9°	83.4°	90°
$M_{S,\min} = M_{T,\min}$	5.00	1.89	1.15	1.00
$\|S\|_\infty$	7.00	2.60	1.59	1.98
$\|T\|_\infty$	7.40	2.76	1.60	1.31

$$W_u = I,\ W_P = \left(\frac{s/M + \omega_B^*}{s}\right) I,\ M = 2,\ \omega_B^* = 0.5$$

The weight W_P indicates that we require $\|S\|_\infty$ less than 2, and require tight control up to a frequency of about $\omega_B^* = 0.5$ rad/s. The corresponding responses to a step change in the reference $r = [1 \quad -1]$, are shown in Figure 3-7.

Several things about the example are worth noting:

1- We see from the simulation for $\phi = \alpha = 0^\circ$ in Figure 3-7 that the response for y_1 is very poor. This is expected because of the closeness of the RHP-pole and zero $(z = 2,\ p = 3)$.

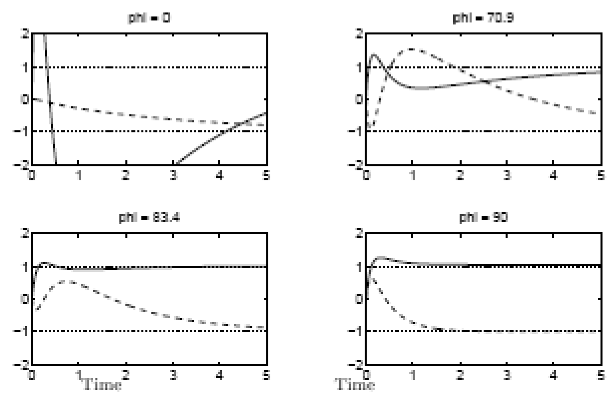

Figure 3-7 MIMO plant with angle ϕ between RHP-pole and RHP-zero. Response to step in reference $r = [1 \quad -1]$ with H_∞ controller for four different values of ϕ Solid line: y_1; Dashed line: y_2

The response for y_2 is also relatively sluggish, because the H_∞ is only concerned with the worth-case response in y_1. The response for y_2 may therefore be faster, if desired.

2- For $\phi = \alpha = 90^\circ$ the RHP-pole and RHP-zero do not interact. From the simulation we see that y_1 (solid line) has an overshoot due to the RHP-pole, whereas y_2 (dashed line) has an undershoot due to the RHP-zero.

3- The lower bound $M_{S,\min} = M_{T,\min}$ on $\|S\|_\infty$ and $\|T\|_\infty$, (see 3-23), is tight in the sense that there exist a controller that achieves it. This can be shown numerically by selecting $W_u = 0.01I$, , $\omega_B^* = 0.01$ and $M = 1$. W_u and ω_B^* are small so the main objective is to minimize the peak of *S*. We find with these weights that the H_∞ designs for the four angles yield $\|S\|_\infty = 5.04,\ ,1.905,\ 1.155,\ 1.005$, which are very close to $M_{S,\min}$.

4- The angle ϕ between the pole and zero is quite different from the rotation angle at intermediate values between 0° and 90°. This is because of the influence of the RHP-pole in output 1, which yields a strong gain in this direction, and thus tends to push the zero direction towards output 2.

5- The H_∞ optimal controller is unstable for $\alpha = 0^\circ$ and 30°. This is not altogether surprising, because for $\alpha = 0^\circ$ the plant becomes two SISO systems one of which needs an unstable controller to stabilize it since $p > z$

3-5 Functional Controllability

Consider a plant *G(s)* with *l* outputs and let *r* denote the normal rank of *G(s).* In order to control all outputs independently we must require $r = l$ and the plant must be "functionally controllable". This term was introduced by Rosenbrock for square systems, and related concepts are "right invertibility" and "output realizability". We will use the following definition:

Definition 3-1 Functional controllability.

An *m*-input *l*-output system *G(s)* is functionally controllable if the normal rank of *G(s),* denoted *r*, is equal to the number of outputs $(r = l)$; that is, if *G(s)* has full row rank. A plant is functionally uncontrollable if $r < l$.

The normal rank of *G(s)* is the rank of *G(s)* at all values of *s* except at a finite number of singularities (which are the zeros of *G(s)*). The minimal requirement for functional controllability is that we have at least many inputs as outputs, i.e. $m \geq l$.

A plant is *functionally uncontrollable* if and only if $\sigma_l(G(j\omega)) = 0,\ \forall\omega$. As a measure of how close a plant is to being functional uncontrollable we may therefore consider the minimum singular value $\sigma_l(G(j\omega))$. The only example of a SISO plant which is functionally uncontrollable is $G(s)=0$. Similarly, a MIMO plant is functionally uncontrollable if the gain is identically zero in some output direction at all frequency.

For strictly proper plants, $G(s)=C(sI-A)^{-1}B$, we have that $G(s)$ is functionally uncontrollable if $rank(B)<l$ (the system is input deficient), or if $rank(C)<l$ (the system is output deficient), or if $rank(sI-A)<l$ (fewer states than outputs). This follows since the rank of a product of matrices is less than or equal to the minimum rank of the individual matrices.

In most cases functional uncontrollability is a structural property of the plant; that is, it does not depend on specific parameter values, and it may often be evaluated from cause-and-effect graphs. A typical example of this is when none of the inputs u_i affect a particular output y_i which would be the case if one of the rows in $G(s)$ was identically zero. Another example is when there are fewer inputs than outputs.

If the plant is not functionally controllable, i.e. $r<l$ then there are $l-r$ output directions, denoted y_0 which cannot be affected. These directions will vary with frequency, and we have (analogous to the concept of a zero direction)

$$y_0^H(j\omega)G(j\omega)=0 \tag{3-26}$$

From an SVD of $G(j\omega)=Y\Sigma U^H$ the uncontrollable output directions $y_0(j\omega)$ are the last $l-r$ columns of $Y(j\omega)$. By analyzing these directions, an engineer can then decide on whether it is acceptable to keep certain output combinations uncontrolled, or if additional actuators are needed to increase the rank of $G(s)$.

Example 3-5

The following plant is singular and thus not functionally controllable

$$G(s)=\begin{bmatrix} \frac{1}{s+1} & \frac{2}{s+1} \\ \frac{2}{s+2} & \frac{4}{s+2} \end{bmatrix}$$

This is easily seen since column 2 of $G(s)$ is two times column 1.The uncontrollable output directions at low and high frequencies are, respectively,

$$y_0(0) = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}, \qquad y_0(\infty) = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

3-6 Limitations Imposed by Time Delays

Time delays pose limitations in MIMO systems, but there are exceptions. As an example of a limitation, let θ_{ij} denote the time delay in the ij'th element of $G(s)$. Then a lower bound on the time delay for output *i* is given by the smallest delay in row *i* of $G(s)$, that is:

$$\theta_i^{\min} = \min_j \theta_{ij}$$

This bound is obvious since $\theta_i^{\min}$ is the minimum time for any input to affect output *i*, and $\theta_i^{\min}$ can be regarded as a delay pinned to output *i*.

For MIMO systems we have the surprising result that an increased time delay may sometimes improve the achievable performance. As a simple example, consider the plant

$$G(s) = \begin{bmatrix} 1 & 1 \\ e^{-\theta s} & 1 \end{bmatrix}$$

With $\theta = 0$ the plant is singular (not functionally controllable) and controlling the two outputs independently is clearly impossible. On the other hand, for $\theta > 0$ affective feedback control is possible at high frequency, provided the bandwidth is larger than about $1/\theta$. That is, for this example control is easier the larger θ is. In words, the presence of the delay decouples the initial (high-frequency) response, so we can obtain tight control if the controller reacts within this initial time period. To illustrate this, we may compute the singular values of G as a function of frequency, and note that the minimum singular value is 0 at low frequencies, but increase with frequency and attains a maximum value of 1.41 at frequency π/θ.

3-7 Limitations Imposed by RHP Zeros

RHP-zeros are common in many practical multivariable problems. The limitations of a RHP-zero located at *z* may also be derived from the bound (by maximum module theorem)

$$\|w_P(s)S(s)\|_\infty = \max_\omega |w_P(j\omega)|.\bar{\sigma}(S(j\omega)) \geq |w_P(z)| \quad 3\text{-}27$$

where $w_P(s)$ is a scalar weight. As usual, we select $1/|w_P|$ as an upper bound on the sensitivity function (see 3-6) so we need $\|w_P(s)S(s)\|_\infty < 1$so to be able to satisfy 3-6 we must at least require that the weight satisfies

$$|w_P(z)| < 1 \quad 3\text{-}28$$

We will now use 3-28 to gain insight into the limitations imposed by RHP-zeros, first by considering a weight that requires good performance at low frequencies, and then by considering a weight that requires good performance at high frequencies.

3-7-1 Performance at Low Frequencies

Consider again the performance weight 3-16

$$w_P(s) = \frac{s/M + \omega_B^*}{s + \omega_B^* A}$$

It specifies a minimum bandwidth ω_B^* (actually $\omega_B^*$is the frequency where the straight-line approximation of the weight crosses 1), a maximum peak of $\bar{\sigma}(S)$less than M and a steady-state offset less than $A < 1$, and at frequencies lower than the bandwidth the sensitivity is required to improve by at least 20 dB/decade (i.e. $\bar{\sigma}(S)$ has slope 1 or larger on a log-log plot). If the plant has a RHP-zero at $s = z$ then from 3-28 we must require

$$|w_P(z)| = \left| \frac{z/M + \omega_B^*}{z + \omega_B^* A} \right| < 1 \quad 3\text{-}29$$

Real zero: Consider the case when z is real. Then all variables in 3-29 are real and positive and 3-29 is equivalent to

$$\omega_B^*(1-A) < z(1 - \frac{1}{M}) \quad 3\text{-}30$$

For example, with $A = 0$ (no steady-state offset) and $M = 2$ ($\bar{\sigma}(S) < 2$) we must at least require

$$\omega_B^* < \frac{z}{2} \quad 3\text{-}31$$

Imaginary zero: For a RHP-zero on the imaginary axis $z = j|z|$ a similar derivation with $A = 0$ yields

$$\omega_B^* < |z|\sqrt{1-\frac{1}{M^2}} \quad \overset{If\ M=2}{\Rightarrow} \quad \omega_B^* < 0.87|z| \qquad 3\text{-}32$$

3-7-2 Performance at High Frequencies

The bounds 3-30 and 3-32 derived above assume tight control at low frequencies. Here, we consider a case where we want tight control at high frequencies, by use of the performance weight

$$w_P(s) = \frac{1}{M} + \frac{s}{\omega_B^*}$$

This requires tight control $\bar{\sigma}(S) < 1$ at frequencies higher than ω_B^* whereas the only requirement at low frequencies is that the peak of $\bar{\sigma}(S)$ is less than *M*. Admittedly, the weight is unrealistic in that it requires $\bar{\sigma}(S) \to 0$ at high frequencies. In any case, to satisfy $\|w_P S(s)\|_\infty < 1$ we must at least require that the weight satisfies 3-28 and with a real RHP-zero we derive for the above weight

$$\omega_B^* > z\frac{1}{1-1/M} \qquad 3\text{-}33$$

For example, with $M = 2$ the requirement is $\omega_B^* > 2z$.

3-7-3 Moving the Effect of a RHP-zero to a Specific Output

In MIMO systems, one can often move the deteriorating effect of a RHP-zero to a less important output. This is possible because, although the interpolation constraint $y_z^H T(z) = 0$ imposes a certain relationship between the elements within each column of $T(s)$, the columns of $T(s)$ may still be selected independently. Let us first consider an example to motivate the results that follow.

Example 3-6

Consider the plant

$$G(s) = \frac{1}{(0.2s+1)(s+1)}\begin{bmatrix} 1 & 1 \\ 1+2s & 2 \end{bmatrix}$$

which has a RHP-zero at $s = z = 0.5$. The output zero direction satisfies $y_z^H G(z) = 0$ and we find

$$y_z = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$$

Any allowable $T(s)$ must satisfy the interpolation constraint $y_z^H T(z) = 0$ and this imposes the following relationships between the column elements of $T(s)$:

$$2t_{11}(z) - t_{21}(z) = 0; \quad 2t_{12}(z) - t_{22}(z) = 0 \qquad \text{3-34}$$

We will consider reference tracking $y = Tr$ and examine three possible choices for T : T_0 diagonal (a decoupled design), T_1 with output 1 perfectly controlled, and T_2 with output 2 perfectly controlled. Of course, we cannot achieve perfect control in practice, but we make the assumption to simplify our argument. In all three cases, we require perfect tracking at steady-state, i.e. $T(0) = I$.

A decoupled design has $t_{12}(s) = t_{21}(s) = 0$ and to satisfy 3-34 we then need $t_{11}(z) = 0$ and $t_{22}(z) = 0$ so the RHP-zero must be contained in both diagonal elements. One possible choice, which also satisfies $T(0) = I$ is

$$T_0(s) = \begin{bmatrix} \frac{-s+z}{s+z} & 0 \\ 0 & \frac{-s+z}{s+z} \end{bmatrix}$$

For the two designs with one output perfectly controlled we choose

$$T_1(s) = \begin{bmatrix} 1 & 0 \\ \frac{\beta_1 s}{s+z} & \frac{-s+z}{s+z} \end{bmatrix} \qquad T_2(s) = \begin{bmatrix} \frac{-s+z}{s+z} & \frac{\beta_2 s}{s+z} \\ 0 & 1 \end{bmatrix}$$

The basis for the last two selections is as follows. For the output which is not perfectly controlled, the diagonal element must have a RHP-zero to satisfy 3-34 and the off-diagonal element must have an s term in the numerator to give $T(0) = I$. To satisfy 3-34 we must then require for the two designs

$$\beta_1 = 4 \qquad \beta_2 = 1$$

The RHP-zero has no effect on output 1 for design $T_1(s)$ and no effect on output 2 for designing $T_2(s)$. We therefore see that it is indeed possible to move the effect of the RHP-zero to a particular output. However, we must pay for this by having to accept some interaction. We note that the magnitude of the interaction, as expressed by β_k, is largest for the case where output 1 is perfectly controlled ($\beta_1 = 4$). This is reasonable since the zero output direction $y_z = \begin{bmatrix} 0.89 \\ -0.45 \end{bmatrix}$ is mainly in the direction of output 1 so we have to "pay more" to push its effect to output 2.

We see from the above example that by requiring a decoupled response from *r* to *y*, as in design $T_0(s)$, we have to accept that the multivariable RHP-zero appears as a RHP-zero in each of the diagonal elements of $T(s)$, i.e. whereas $G(s)$ has one RHP-zero at $s = z$, $T_0(s)$ has two. In other words, requiring a decoupled response generally leads to the introduction of additional RHP zeros in $T(s)$ which are not present in the plant $G(s)$.

We also see that we can move the effect of the RHP-zero to a particular output, but we then have to accept some interaction. This is stated more exactly in the following Theorem.

Theorem 3-2

Assume that $G(s)$ is square, functionally controllable and stable and has a single RHP-zero at $s = z$ and no RHP-pole at $s = z$. Then if the *k*'th element of the output zero direction is non-zero, i.e. $y_{zk} \neq 0$, it is possible to obtain "perfect" control on all outputs $j \neq k$ with the remaining output exhibiting no steady-state offset. Specifically, *T* can be chosen of the form

$$T(s) = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & . & . & . & \cdots & . \\ \frac{\beta_1 s}{s+z} & \frac{\beta_2 s}{s+z} & \cdots & \frac{\beta_{k-1} s}{s+z} & \frac{-s+z}{s+z} & \frac{\beta_{k+1} s}{s+z} & \cdots & \frac{\beta_n s}{s+z} \\ . & . & \cdots & . & . & . & \cdots & . \\ . & . & \cdots & . & . & . & \cdots & . \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \qquad \text{3-35}$$

where

$$\beta_j = -2\frac{y_{zj}}{y_{zk}} \quad for \;\; j \neq k \qquad \text{3-36}$$

Proof: It is clear that 3-35 satisfies the interpolation constraint $y_z^H T(z) = 0$; see also Holt and Morari (1985)

The effect of moving completely the effect of a RHP-zero to output k is quantified by 3-36. We see that if the zero is not "naturally" aligned with this output, i.e. if $|y_{zk}|$ is much smaller than 1 then the interactions will be significant, in terms of yielding some $\beta_j = -2\frac{y_{zj}}{y_{zk}}$ much larger than 1 in magnitude. In particular, we cannot move the effect of a RHP-zero to an output corresponding to a zero element in y_z, which occurs frequently if we have a RHP-zero pinned to a subset of the outputs.

3-8 Limitations Imposed by Unstable (RHP) Poles

For unstable plants we need feedback for stabilization. The limitations of a RHP-pole located at *p* may also be derived from the bound (by maximum module theorem)

$$\|w_T(s)T(s)\|_\infty = \max_\omega |w_T(j\omega)|.\bar{\sigma}(T(j\omega)) \geq |w_T(p)| \tag{3-37}$$

Consider that the weight $w_T(s)$ is selected such that $1/|w_T|$ is a reasonable upper bound on the complementary sensitivity function so we need $\|w_T(s)T(s)\|_\infty < 1$. This condition and 3-37 at least require that the weight satisfies

$$|w_T(p)| < 1 \tag{3-38}$$

Now consider the following weight

$$w_T(s) = \frac{s}{\omega_{BT}^*} + \frac{1}{M_T} \tag{3-39}$$

which requires T to have a roll-off rate of at least 1 at high frequencies (which must be satisfied for any real system), that $\bar{\sigma}(T)$ is less than M_T at low frequencies, and that $\bar{\sigma}(T)$ drops below 1 at frequency ω_{BT}^*. The requirement on $\bar{\sigma}(T)$ ($|T|$ in SISO case) is shown graphically in Figure 3-8

Real RHP-pole at $s = p$. For the weight 3-39 condition 3-38 yields

$$\omega_{BT}^* > p\frac{M_T}{M_T - 1} \quad 3\text{-}40$$

Thus, the presence of the RHP-pole puts a lower limit on the bandwidth in terms of *T*; that is, we cannot let the system roll-off at frequencies lower than *p*. For example, with $M_T = 2$, we get $\omega_{BT}^* > 2p$ which is approximately achieved if

$$\omega_c > 2p \quad 3\text{-}41$$

Imaginary RHP-pole. For a purely imaginary pole located at $p = j|p|$ a similar analysis of the weight 3-39 with $M_T = 2$ shows that we must at least require $\omega_{BT}^* > 1.15|p|$ which is approximately achieved if

$$\omega_c > 1.15|p| \quad 3\text{-}42$$

In conclusion, we find that stabilization with reasonable performance requires a bandwidth which is larger than the distance $|p|$ of the RHP-pole from the origin.

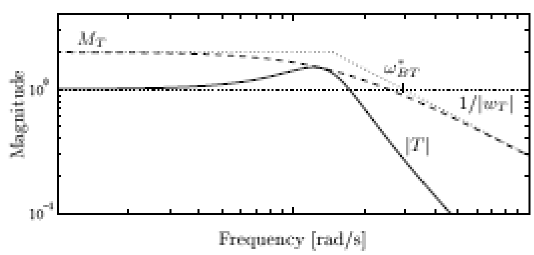

Figure 3-8 Typical complementary sensitivity, $|T|$, with upper bound $1/|w_T|$

Exercises

3-1 Derive equations 3-31 and 3-32.

3-2 Consider the weight

$$w_P(s) = \frac{s + M\omega_B^*}{s} \frac{s + fM\omega_B^*}{s + fM^2\omega_B^*}$$

with $f > 1$. This is the same weight as 3-16 with $A = 0$ except that it approaches 1at high frequencies, and *f* gives the frequency range over which we allow a peak. Plot the weight for $f = 10$ and $M = 2$. Derive an upper bound on ω_B^* for the case with $f = 10$ and $M = 2$.

3-3 Consider the weight $w_P(s) = \frac{1}{M} + \left(\frac{\omega_B^*}{s}\right)^n$ which requires $|S|$ to have a slope of *n* at low frequencies and requires its low-frequency asymptote to cross 1at a frequency ω_B^*. Note that $n = 1$ yields the weight 3-16 with $A = 0$. Derive an upper bound on ω_B^* when the plant has a RHP-zero at *z*. Show that the bound becomes $\omega_B^* \le |z|\ as\ n \to \infty$.

$$w_P(s) = \frac{\left(\frac{100s}{\omega_B^*} + \frac{1}{M}\right)\left(\frac{s}{M\omega_B^*} + 1\right)}{\left(\frac{10s}{\omega_B^*} + 1\right)\left(\frac{100s}{\omega_B^*} + 1\right)}$$

3-4 Consider the case of a plant with a RHP zero where we want to limit the sensitivity function over some frequency range. To this effect let

This weight is equal to $1/M$ at low and high frequencies, has a maximum value of about $10/M$ at intermediate frequencies, and the asymptote crosses 1 at frequencies $\omega_B^*/1000$ and ω_B^*. Thus we require "tight" control, $|S| < 1$, in the frequency range between $\omega_{BL}^* = \omega_B^*/1000$ and $\omega_{BH}^* = \omega_B^*$.

a) Make a sketch of $1/|w_P|$ (which provides an upper bound on $|S|$).

b) Show that the RHP-zero cannot be in the frequency range where we require tight control, and that we can achieve tight control either at frequencies below about $z/2$ (the usual case) or above about $2z$. To see this select $M = 2$ and evaluate $w_p(z)$ for various values of $\omega_B^* = kz$, e.g., $k = 0.1, 0.5, 1, 10, 100, 1000, 2000, 10000$. (you will find that $w_p(z) = 0.95(\approx 1)$ for $k = 0.5$ (corresponding to the requirement $\omega_{BH}^* < z/2$) and for $k = 2000$ (corresponding to the requirement $\omega_{BL}^* > 2z$)).

3-5 Consider the plant

$$G(s) = \begin{bmatrix} \alpha & 1 \\ \frac{1}{s+1} & \alpha \end{bmatrix}$$

a) Find the zero and its output direction. (Answer $z = \frac{1}{\alpha^2} - 1$ and $y_z = \begin{bmatrix} -\alpha \\ 1 \end{bmatrix}$)

b) Which values of α yield a RHP-zero, and which of these values is best/worst in terms of achievable performance? (Answer: We have a RHP-zero for $|\alpha| < 1$. Best for $\alpha = 0$ with zero at infinity: if control at steady-state required then worst for $\alpha = 1$ with zero at $s = 0$.)

c) Suppose $\alpha = 0.1$. Which output is the most difficult to control? Illustrate your conclusion using Theorem 3-2. (Answer: Output 2 is the most difficult since the zero is mainly in that direction; we get interaction $\beta = 20$ if we want to control y_2 perfectly.

3-6 Repeat the exercise 3-5 for the plant

$$G(s) = \frac{1}{s+1} \begin{bmatrix} s-\alpha & 1 \\ (\alpha+2)^2 & s-\alpha \end{bmatrix}$$

3-7 Derive the bound in 3-42

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