Lectures on Multivariable systems

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Stability Analysis in Multivariable Systems

This chapter introduces the feedback structure and discusses its stability properties. The arrangement of this chapter is as follows: Section 3-1 introducing feedback structure and describes the general feedback configuration and the well-posedness of the feedback loop is defined. Next, the notion of internal stability is introduced and the relationship is established between the state space characterization of internal stability and the transfer matrix characterization of internal stability in section 3-2. The stable coprime factorizations of rational matrices are also introduced in section 3-3. Section 3-4 discusses how to achieve a stabilizing controller.

3-1 Well-Posedness of Feedback Loop

We will consider the standard feedback configuration shown in figure 3.1. It consists of the interconnected plant P and controller K forced by command r, sensor noise n, plant input disturbance d_i, and plant output disturbance d. In general, all signals are assumed to be multivariable, and all transfer matrices are assumed to have appropriate dimensions.

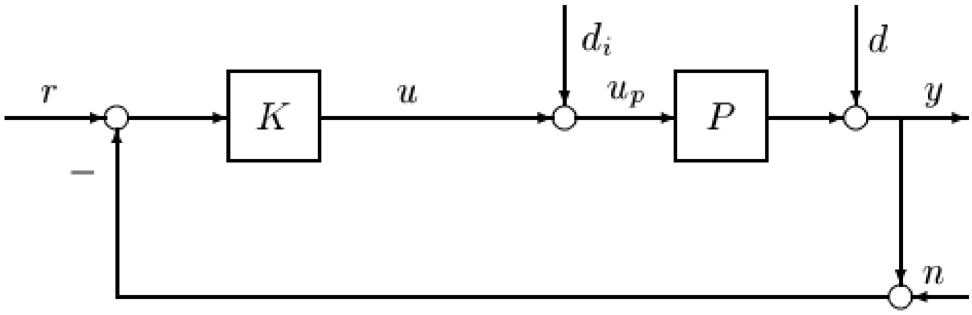

Figure 3.1 Standard feedback configuration

Assume that the plant P and the controller K in Figure 3.1 are fixed real rational proper transfer matrices. Then the first question one would ask is whether the feedback interconnection makes sense or is physically realizable. To be more specific, consider a simple example where

$$P = -\frac{s-1}{s+2}, \quad K = 1$$

are both proper transfer functions. However,

$$u = \frac{s+2}{3}(r-n-d) - \frac{s-1}{3}d_i$$

i.e., the transfer functions from the external signals r, n, d and d_i to u are not proper. Hence, the feedback system is not physically realizable!

Definition 3.1 A feedback system is said to be well-posed if all closed-loop transfer matrices are well-defined and proper.

Now suppose that all the external signals r, n, d and d_i are specified and that the closed-loop transfer matrices from them to u are respectively well-defined and proper. Then, y and all other signals are also well-defined and the related transfer matrices are proper. Furthermore, since the transfer matrices from d and n to u are the same and differ from the transfer matrix from r to u by only a sign, the system is well-posed if and only if the transfer matrix from , d_i , d, to u exists and is proper.

Lemma 3.1 The feedback system in figure 3.1 is well-posed if and only if

$$I + K(\infty)P(\infty) \qquad 3\text{-}1$$

is invertible.

Proof. As we explain the system is well-posed if and only if the transfer matrix from, d_i , d, to u exists and is proper. The transfer matrix from d_i , d, to u is:

$$u = -(I + KP)^{-1}\begin{bmatrix} K & KP \end{bmatrix}\begin{bmatrix} d \\ d_i \end{bmatrix}$$

Thus well-posedness is equivalent to the condition that $(I + KP)^{-1}$ exists and is proper. But this is equivalent to the condition that the constant term of the transfer matrix $I + K(\infty)P(\infty)$ is invertible.

It is straightforward to show that (3.1) is equivalent to either one of the following two conditions:

$$\begin{bmatrix} I & K(\infty) \\ -P(\infty) & I \end{bmatrix} \text{ is invertible} \qquad 3\text{-}2$$

$I + K(\infty)P(\infty)$ is invertible

The well-posedness condition is simple to state in terms of state-space realizations. Introduce realizations of P and K:

$$P \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

$$K \cong \begin{bmatrix} \hat{A} & \hat{B} \\ \hat{C} & \hat{D} \end{bmatrix} \qquad 3\text{-}3$$

Then $P(\infty) = D$ and $K(\infty) = \hat{D}$. For example, well-posedness in (3-2) is equivalent to the condition that

$$\begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix} \text{ is invertible} \qquad 3\text{-}4$$

Fortunately, in most practical cases we will have $D = 0$, and hence well-posedness for most practical control systems is guaranteed.

3-2 Internal Stability

Consider a system described by the standard block diagram in figure 3.1 and assume the system is well-posed. Furthermore, assume that the realizations for *P(s)* and *K(s)* given in equation (3-3) are stabilizable and detectable.

Let x and $\hat{x}$ denote the state vectors for *P* and *K*, respectively, and write the state equations in figure 3.1 with d, d_i and n set to zero:

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \\ \dot{\hat{x}} &= \hat{A}\hat{x} - \hat{B}y \\ u &= \hat{C}\hat{x} - \hat{D}y \end{aligned} \tag{3-5}$$

Definition 3.2 The system of figure 3.1 is said to be internally stable if the origin $(x, \hat{x}) = (0, 0)$ is asymptotically stable, i.e., the states $(x, \hat{x})$ go to zero from all initial states when $d=0$, $d_i=0$ and $n=0$.

Note that internal stability is a state space notion. To get a concrete characterization of internal stability, solve equations (3-5) for y and u:

$$\begin{bmatrix} u \\ y \end{bmatrix} = \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{3-6}$$

Note that the existence of the inverse is guaranteed by the well-posedness condition. Now substitute this into (3-5) and (5-8) to get

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ \hat{x} \end{bmatrix} = \tilde{A} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \tag{3-7}$$

Where

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \tag{3-8}$$

Thus internal stability is equivalent to the condition that $\tilde{A}$ has all its eigenvalues in the open left-half plane. In fact, this can be taken as a definition of internal stability.

Lemma 3.2 The system of figure 3.1 with given stabilizable and detectable realizations for *P* and *K* is internally stable if and only if $\tilde{A}$ is a Hurwitz matrix.

It is routine to verify that the above definition of internal stability depends only on P and K, not on specific realizations of them as long as the realizations of P and K are both stabilizable and detectable, i.e., no extra unstable modes are introduced by the realizations.

The above notion of internal stability is defined in terms of state-space realizations of P and K. It is also important and useful to characterize internal stability from the transfer matrix point of view. Note that the feedback system in figure 3.1 is described, in term of transfer matrices, by

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} d_i \\ -r \end{bmatrix} \qquad 3\text{-}9$$

Note that we ignore d and n as inputs since they produce similar transfer matrices as equation 3-9. Now it is intuitively clear that if the system in figure 3.1 is internally stable, then for all bounded inputs (d_i, -r), the outputs (u_p, - e) are also bounded. The following lemma shows that this idea leads to a transfer matrix characterization of internal stability.

Lemma 3.3 The system in figure 3.1 is internally stable if and only if the transfer matrix

$$\begin{bmatrix} I & K \\ -P & I \end{bmatrix}^{-1} = \begin{bmatrix} I - K(I+PK)^{-1}P & -K(I+PK)^{-1} \\ (I+PK)^{-1}P & (I+PK)^{-1} \end{bmatrix} \qquad 3\text{-}10$$

from (d_i, -r) to (u_p, - e) be a proper and stable transfer matrix.

Proof. Let stabilizable and detectable realizations of P and K defined as 3-3. Then we have the state space equation for the system in figure 3.1 is

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix}\begin{bmatrix} u_p \\ -e \end{bmatrix}$$

$$\begin{bmatrix} y \\ u \end{bmatrix} = \begin{bmatrix} C & 0 \\ 0 & \hat{C} \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} D & 0 \\ 0 & \hat{D} \end{bmatrix}\begin{bmatrix} u_p \\ e \end{bmatrix}$$

$$\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}\begin{bmatrix} y \\ u \end{bmatrix} + \begin{bmatrix} d_i \\ -r \end{bmatrix}$$

The deleting $\begin{bmatrix} y \\ u \end{bmatrix}$ from last two equations we can rewritten a

$$\begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} d_i \\ -r \end{bmatrix} \quad \Rightarrow \quad \begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}\begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1}\begin{bmatrix} d_i \\ -r \end{bmatrix}$$

By substituting this in the states space equation we have

$$\begin{bmatrix} \dot{x} \\ \dot{\hat{x}} \end{bmatrix} = \left(\begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right) \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} u_p \\ -e \end{bmatrix}$$

$$\begin{bmatrix} d_i \\ -r \end{bmatrix} = \begin{bmatrix} 0 & -\hat{C} \\ -C & 0 \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} + \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix} \begin{bmatrix} u_p \\ -e \end{bmatrix}$$

Now suppose that this system is internally stable. So the eigenvalues of

$$\tilde{A} = \begin{bmatrix} A & 0 \\ 0 & \hat{A} \end{bmatrix} + \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1} \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix}$$

are in the open left-half plane, it follows that the transfer matrix from (d_i, -r) to (u_p,- e) given in (3-10) is stable.

Conversely, suppose that $(I + PK)$ is invertible and the transfer matrix in (3-10) is stable. Then, in particular, $(I + PK)^{-1}$ is proper which implies that $(I + P(\infty)K(\infty)) = (I + D\hat{D})$ is invertible. Therefore,

$$\begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}$$

is nonsingular. Now routine calculations give the transfer matrix from (d_i, -r) to (u_p, -e) in terms of the state space realizations:

$$\begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix} + \begin{bmatrix} 0 & -\hat{C} \\ -C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & -\hat{B} \end{bmatrix} \begin{bmatrix} I & \hat{D} \\ -D & I \end{bmatrix}^{-1}$$

Since the above transfer matrix is stable, it follows that

$$\begin{bmatrix} 0 & -\hat{C} \\ -C & 0 \end{bmatrix} (sI - \tilde{A})^{-1} \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}$$

as a transfer matrix is stable. Finally, since (A, B, C) and $(\hat{A}, \hat{B}, \hat{C})$ are stabilizable and detectable,

$$\left(\tilde{A}, \begin{bmatrix} B & 0 \\ 0 & \hat{B} \end{bmatrix}, \begin{bmatrix} 0 & \hat{C} \\ C & 0 \end{bmatrix} \right)$$

is stabilizable and detectable. It then follows that the eigenvalues of $\tilde{A}$ are in the open left-half plane.

Note that to check internal stability, it is necessary (and sufficient) to test whether each of the four transfer matrices in (3-10) is stable. Stability cannot be concluded even if three of the four transfer matrices in (3-10) are stable. For example, let an interconnected system transfer function be given by

$$P = \frac{s-1}{s+1}, \quad K = \frac{1}{s-1}$$

Then it is easy to compute

$$\begin{bmatrix} u_p \\ -e \end{bmatrix} = \begin{bmatrix} \dfrac{s+1}{s+2} & -\dfrac{s+1}{(s-1)(s+2)} \\ \dfrac{s-1}{s+2} & \dfrac{s+1}{s+2} \end{bmatrix} \begin{bmatrix} d_i \\ -r \end{bmatrix}$$

which shows that the system is not internally stable although three of the four transfer functions are stable. This can also be seen by calculating the closed-loop A-matrix with any stabilizable and detectable realizations of *P* and *K*.

.*Remark 3.1* It should be noted that internal stability is a basic requirement for a practical feedback system. This is because all interconnected systems may be unavoidably subject to some nonzero initial conditions and some (possibly small) errors, and it cannot be tolerated in practice that such errors at some locations will lead to unbounded signals at some other locations in the closed-loop system. Internal stability guarantees that all signals in a system are bounded provided that the injected signals (at any locations) are bounded. However, there are some special cases under which determining system stability is simple.

Corollary 3.4 Suppose K is stable. Then the system in figure 3.1 is internally stable iff $(I+PK)^{-1}P$ is stable.

Proof. The necessity is obvious. To prove the sufficiency, it is sufficient to show that if $Q=(I+PK)^{-1}P$ is stable, the other three transfer matrices are also stable since:

$$I - K(I+PK)^{-1}P = I - KQ$$
$$-K(I+PK)^{-1} = -K(I-QK)$$
$$(I+PK)^{-1} = I + (I+PK)^{-1} - I = I - (I+PK)^{-1}PK = I - QK$$

This corollary is in fact the basis for the classical control theory where the stability is checked only for one closed-loop transfer function with the implicit assumption that the controller itself is stable. Also, we have

Corollary 3.5 Suppose P is stable. Then the system in figure 3.1 is internally stable iff $K(I+PK)^{-1}$ is stable.

Proof. The necessity is obvious. To prove the sufficiency, it is sufficient to show that if $Q = K(I+PK)^{-1}$ is stable, the other three transfer matrices are also stable since:

$$I - K(I+PK)^{-1}P = I - QP$$
$$(I+PK)^{-1}P = (I-PQ)P$$
$$(I+PK)^{-1} = I + (I+PK)^{-1} - I = I - PK(I+PK)^{-1} = I - PQ$$

Corollary 3.6 Suppose P and K are both stable. Then the system in figure 3.1 is internally stable iff $(I+PK)^{-1}$ is stable.

Proof. The necessity is obvious. To prove the sufficiency, it is sufficient to show that if $Q = (I+PK)^{-1}$ is stable, the other three transfer matrices are also stable since:

$$I - K(I+PK)^{-1}P = I - KQP$$
$$(I+PK)^{-1}P = QP$$
$$K(I+PK)^{-1} = KQ$$

To study the more general case, define

n_c = number of open RHP poles of $K(s)$

n_p = number of open RHP poles of $P(s)$

Theorem 3.7 The system in figure 3.1 is internally stable if and only if

(i) the number of open RHP poles of $P(s)\,K(s)$ = $n_c + n_p$(it simply means that there is no RHP pole zero cancellation between plant and controller).

(ii) $\phi(s) = \det(I + P(s)K(s))$ has all its zeros in the open left-half plane (i.e., $(I + P(s)K(s))^{-1}$ is stable).

Proof. See "Robust and Optimal Control By Kemin Zhou"

Theorem 3.8 (Nyquist Stability Theorem) The system in figure 3.1 is internally stable if and only if condition (i) in Theorem 3.8 is satisfied and the Nyquist plot of $\phi(j\omega)$ for $-\infty \le \omega \le \infty$ encircles the origin, $n_c + n_p$ times in the counter-clockwise direction.
Proof. See "Robust and Optimal Control By Kemin Zhou"

3-3 Coprime Factorization over stable transfer functions

Recall that two polynomials *m(s)* and *n(s)*, with, for example, real coefficients, are said to be coprime if their greatest common divisor is 1 (equivalent, they have no common zeros). It follows from Euclid's algorithm that two polynomials *m* and *n* are coprime if there exist polynomials x(s) and *y(s)* such that $xm + yn = 1$; such an equation is called a Bezout identity. Similarly, two transfer functions *m(s)* and *n(s)* in the set of stable transfer functions are said to be coprime over stable transfer functions if there exists x, y in the set of stable transfer functions such that

$$xm + yn = 1$$

More generally, we have

Definition 3.3 Two matrices M and N in the set of stable transfer matrices are right coprime over the set of stable transfer matrices if they have the same number of columns and if there exist matrices X_r and Y_r in the set of stable transfer matrices such
That

$$\begin{bmatrix} X_r & Y_r \end{bmatrix} \begin{bmatrix} M \\ N \end{bmatrix} = X_r M + Y_r N = I$$

Similarly, two matrices $\tilde{M}$ and $\tilde{N}$ in the set of stable transfer matrices are left coprime over the set of stable transfer matrices if they have the same number of rows and if there exist two matrices X_l and Y_l in the set of stable transfer matrices such that

$$\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix} \begin{bmatrix} X_l \\ Y_l \end{bmatrix} = \tilde{M} X_l + \tilde{N} Y_l = I$$

Note that these definitions are equivalent to saying that the matrix $\begin{bmatrix} M \\ N \end{bmatrix}$ is left invertible in the set of stable transfer matrices and the matrix $\begin{bmatrix} \tilde{M} & \tilde{N} \end{bmatrix}$ is right-invertible in the set of stable transfer matrices. These two equations are often called Bezout identities.

Now let P be a proper real-rational matrix. A right-coprime factorization (rcf) of P is a factorization $P = NM^{-1}$ where N and M are right-coprime in the set of stable transfer matrices. Similarly, a left-coprime factorization (lcf) has the form $P = \tilde{M}^{-1}\tilde{N}$ where $\tilde{N}$ and $\tilde{M}$ are left-coprime over the set of stable transfer matrices. A matrix $P(s)$ in the set of rational proper transfer matrices is said to have double coprime factorization if there exist a right coprime factorization $P = NM^{-1}$, a left coprime $P = \tilde{M}^{-1}\tilde{N}$ and X_r, Y_r, X_l, Y_l in the set of stable transfer matrices such that

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix}\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} = I \qquad \text{3-11}$$

Of course implicit in these definitions is the requirement that both M and $\tilde{M}$ be square and nonsingular.

Theorem 3.9 Suppose $P(s)$ is a proper real-rational matrix and

$$P(s) \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a stabilizable and detectable realization. Let F and L be such that $A+BF$ and $A+LC$ are both stable, and define

$$\begin{bmatrix} M & -Y_l \\ N & X_l \end{bmatrix} \cong \begin{bmatrix} A+BF & B & -L \\ F & I & 0 \\ C+DF & D & I \end{bmatrix}$$

$$\begin{bmatrix} X_r & Y_r \\ -\tilde{N} & \tilde{M} \end{bmatrix} \cong \begin{bmatrix} A+LC & -(B+LD) & L \\ F & I & 0 \\ C & -D & I \end{bmatrix}$$

Then $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ are rcf and lcf, respectively, and, furthermore, (3.11) is satisfied.

The coprime factorization of a transfer matrix can be given a feedback control interpretation. For example, right coprime factorization comes out naturally from changing the control variable by a state feedback. Consider the state space equations for a plant P in the figure 3.2:

$$\dot{x} + Ax + Bu$$
$$y = Cx + Du$$

Next, introduce a state feedback and change the variable

$$u=v+Fx$$

where F is such that $A + BF$ is stable. Then we get

$$\dot{x} = \{A + BF)x + Bv$$
$$u = Fx + v$$
$$y = (C + DF)x + Dv$$

Evidently from these equations, the transfer matrix from v to u is

$$M(s) \cong \begin{bmatrix} A+BF & B \\ F & I \end{bmatrix}$$

and that from v to y is

$$N(s) \cong \begin{bmatrix} A+BF & B \\ C+DF & D \end{bmatrix}$$

Therefore

$$u = Mv,\ y = Nv$$

so that $y = NM^{-1}u$, i.e., $P = NM^{-1}$.

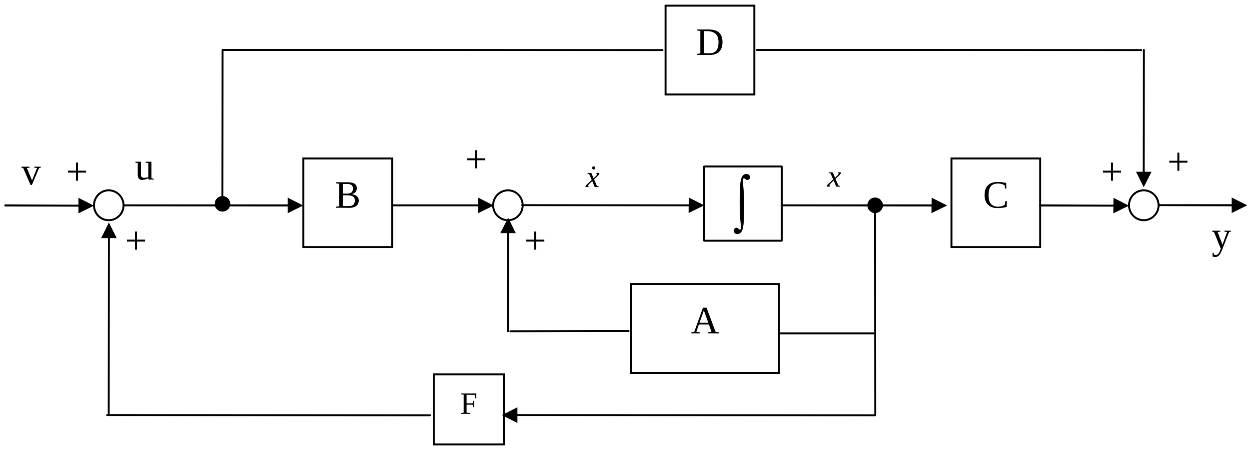

Figure 3.2 Feedback representation of coprime factorization

3-4 Stabilizing controllers

In this section we introduce a parameterization known as the Q-parameterization or Youla-parameterization, of all stabilizing controllers for a plant. By all stabilizing controllers we mean all controllers that yield internal stability of the closed loop system. We first consider stable plants for which the parameterization is easily derived and then unstable plants where we make use of the coprime factorization.

The following lemma forms the basis for parameterizing all stabilizing controllers for stable plants.

Corollary 3.10 Suppose P is stable. Then the set of all stabilizing controllers in figure 3.1 can be described as

$$K = Q(I - PQ)^{-1} \qquad \text{3-12}$$

for any Q in the set of stable transfer matrices and $I - P(\infty)Q(\infty)$ nonsingular.

Proof. First we know that if $I - P(\infty)Q(\infty)$ is nonsingular then $K = Q(I - PQ)^{-1}$ exist and

$$K = Q(I - PQ)^{-1} \quad \Rightarrow \quad K(I - PQ) = Q \quad \Rightarrow \quad Q = K(I + PK)^{-1}$$

Since Q is stable so $K(I + PK)^{-1}$ is stable so by corollary 3.5 in figure 3.1 is stable. In other suppose the system in the figure 3.1 is stable so by corollary 3.5 $K(I + PK)^{-1}$ is stable. Define $Q = K(I + PK)^{-1}$ so (note that $I - P(\infty)Q(\infty)$ is nonsingular)

$K = Q(I - PQ)^{-1}$

Example 3.1: For the plant

$$P(s) = \frac{1}{(s+1)(s+2)}$$

Suppose that it is desired to find an internally stabilizing controller so that *y* asymptotically tracks a ramp input.

Solution: Since the plant is stable the set of all stabilizing controller is derived from

$K = Q(I - PQ)^{-1}$, for any stable Q such that, $I - P(\infty)Q(\infty)$ is nonsingular.

Let

$$Q = \frac{as+b}{s+3}$$

We must define the variables *a* and *b* such that that *y* asymptotically tracks a ramp input, so *S* must have two zeros at origin.

$$S = 1 - T = 1 - PK(I + PK)^{-1} = 1 - PQ = 1 - \frac{as+b}{(s+1)(s+2)(s+3)} = \frac{(s+1)(s+2)(s+3) - as + b}{(s+1)(s+2)(s+3)}$$

So we should take $a = 11,\ b = 6$. This gives

$$Q = \frac{11s+6}{s+3}$$

$$K = \frac{11(s+1)(s+2)(s+12/22)}{s^2(s+6)}.$$

However if *P(s)* is not stable, the parameterization is much more complicated. The result can be more conveniently stated using state-space representations. The following theorem shows that a proper real-rational plant may be stabilized irrespective of the location of its RHP-poles and RHP-zeros, provided the plant does not contain unstable hidden modes.

Corollary 3.11 Let *P* be a proper real-rational matrix and $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be corresponding rcf and lcf over the set of stable transfer matrices. Then there exists a stabilizing controller $K = \tilde{U}\tilde{V}^{-1} = V^{-1}U$ with $U, V, \tilde{U}, \tilde{V}$ in the set of stable transfer matrices ($VM + UN = I$, $\tilde{N}\tilde{U} + \tilde{M}\tilde{V} = I$).

Furthermore, suppose

$$P(s) \cong \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

is a stabilizable and detectable realization of *P* and let *F* and *L* be such that *A+BF* and *A+LC* are stable. Then a particular set of state space realizations for these matrices can be given by

$$\begin{bmatrix} M & -\tilde{U} \\ N & \tilde{V} \end{bmatrix} = \begin{bmatrix} A+BF & B & -L \\ F & I & 0 \\ C+DF & D & I \end{bmatrix}$$

$$\begin{bmatrix} V & U \\ -\tilde{N} & \tilde{M} \end{bmatrix} = \begin{bmatrix} A+LC & -(B+LD) & L \\ F & I & 0 \\ C & -D & I \end{bmatrix}$$

Proof. See "Robust and Optimal Control By Kemin Zhou".

The following theorem forms the basis for parameterizing all stabilizing controllers for a proper real-rational matrix.

Theorem 3.12 Let *P* be a proper real-rational matrix and $P = NM^{-1} = \tilde{M}^{-1}\tilde{N}$ be corresponding rcf and lcf over the set of stable transfer matrices. Then the set of all stabilizing controllers in figure 3.1 can be described as

$$K = (V - Q_l\tilde{N})^{-1}(U + Q_l\tilde{M})$$

Or 3-13

$$K = (\tilde{U} + MQ_r)(\tilde{V} - NQ_r)^{-1}$$

where Q_l is any stable transfer matrices and $V(\infty) - Q_l(\infty)\tilde{N}(\infty)$ is nonsingular or Q_r is any stable transfer matrices and $\tilde{V}(\infty) - N(\infty)Q_r(\infty)$.

Proof. See "Robust Normalized coprime factorization for non-strictly proper systems By Vidyasagar".

Example 3.2: For the plant

$$P(s) = \frac{1}{(s-1)(s-2)}$$

Find a stabilizing controller.

Solution: First of all it is clear that

$$P(s) \cong \left[\begin{array}{ccc} 0 & 1 & 0 \\ -2 & 3 & 1 \\ 1 & 0 & 0 \end{array}\right]$$

is a stabilizable and detectable realization. Now let $F = [1 \quad -5]$ and $L = [-7 \quad -23]^T$ clearly $A+BF$ and $A+LC$ are stable. Since the plant is unstable we use corollary 3.11 to derive coprime factorization parameters.

$$N = \frac{1}{(s+1)^2},\ M = \frac{(s-2)(s-1)}{(s+1)^2},\ \tilde{U} = \frac{108s-72}{(s+1)^2},\ \tilde{V} = \frac{s^2+9s+38}{(s+1)^2}$$

$$\tilde{N} = \frac{1}{(s+2)^2},\ \tilde{M} = \frac{(s-2)(s-1)}{(s+2)^2},\ U = \frac{108s-72}{(s+2)^2},\ V = \frac{s^2+9s+38}{(s+2)^2}$$

Now let $Q_r = Q_l = 0$ then by theorem 3.12 one of the stabilizing controllers is:

$$K = \tilde{U}\tilde{V}^{-1} = \frac{108s-72}{s^2+9s+38}$$

Example 3.3: For the plant in figure 3.1

$$P(s) = \frac{1}{(s-1)(s-2)}$$

The problem is to find a controller that

1. The feedback system is internally stable.
2. The final value of *y* equals 1 when *r* is a unit step and *d=0.*
3. The final value of *y* equals zero when *d* is a sinusoid of 10 rad/s and *r=0.*

Solution: The set of all stabilizing controller is:

$$K = (\tilde{U} + MQ_r)(\tilde{V} - NQ_r)^{-1}$$

where from the example 3.2 we have

$$N = \frac{1}{(s+1)^2},\ M = \frac{(s-2)(s-1)}{(s+1)^2},\ \tilde{U} = \frac{108s-72}{(s+1)^2},\ \tilde{V} = \frac{s^2+9s+38}{(s+1)^2}$$

Clearly for any stable *Q* the condition 1 satisfied.

To met condition 2 the transfer function from *r* to *y* ($N(\tilde{U} + MQ_r)$)must satisfy

$$N(0)(\tilde{U}(0) + M(0)Q_r(0)) = 1 \quad \Rightarrow \quad Q_r(0) = 36.5$$

To met condition 3 the transfer function from *d* to *y* ($M(\tilde{V} - NQ_r)$)must satisfy

$$M(10j)(\tilde{V}(10j) - N(10j)Q_r(10j)) = 0 \qquad \Rightarrow \qquad Q_r(10j) = -62 + 90j$$

Now define

$$Q_r(s) = x_1 + x_2 \frac{1}{s+1} + x_3 \frac{1}{(s+1)^2}$$

And then set x_1, x_2 and x_3 to met the requests.

3-5 Strong and simultaneous stabilization

Practicing control engineers are reluctant to use unstable controllers, especially when the plant itself is stable. Since if a sensor or actuator fails, and the feedback loop opens, overall stability is maintained if both plant and controller individually are stable. If the plant itself is unstable, the argument against using an unstable controller is less compelling. However, knowledge of when a plant is or is not stabilizable with a stable controller is useful for another problem namely, simultaneous stabilization, meaning stabilization of several plants by the same controller.

The issue of simultaneous stabilization arises when a plant is subject to a discrete change, such as when a component burns out. Simultaneous stabilization of two plants can also be viewed as an example of a problem involving highly structured uncertainty.

Say that a plant is strongly stabilizable if internal stabilization can be achieved with a controller itself is a stable transfer matrix. Following theorem shows that poles and zeros of *P* must share a certain property in order for *P* to be strongly stabilizable.

Theorem 3.13 *P* is strongly stabilizable if and only if it has an even number of real poles between every pairs of real RHP zeros(including zeros at infinity).

Proof. See “Linear feedback control By Doyle”.

Example 3.4: Which of the following plant is strongly stabilizable.

$$P_1(s) = \frac{s-1}{s(s-2)} \qquad\qquad P_2(s) = \frac{(s-1)^2(s^2-s+1)}{(s-2)^2(s+1)^3}$$

Solution: P_1 is not strongly stabilizable since it has one pole between $z=1$ and $z=\infty$ but P_2 is strongly stabilizable since it has two pole between $z=1$ and $z=\infty$.

Exercises

3-1 Find two different lcf's for the following transfer matrix.

$$G(s) = \begin{bmatrix} \frac{s-1}{s+1} & \frac{s-2}{s+2} \end{bmatrix}$$

3-2 Find a lcf's and a lcf's for the following transfer matrix.

$$G(s) = \frac{1}{s+2}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

3-3 Find a lcf's and a lcf's for the following transfer matrix.

$$G(s) = \frac{1}{(s+2)(s-3)}\begin{bmatrix} s-1 & 4 \\ 4.5 & 2(s-1) \end{bmatrix}$$

3-4 Derive a feedback control interpretation for the left coprime factorization.

3-5 Show that the equation of all stabilizing controller for stable plants (eq. 3.12) is a special case of the equation of all stabilizing controller for proper real-rational plants (eq. 3.13).

3-6 In example 3.3 find x1, x_2 and x_3.

3-7 In example 3.3 find the controller and then find the step response of the system. Then suppose the input is zero but a sinusoid with frequency of 10 rad/s applied as disturbance, find the response of system.