Eigenstructure Assignment

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Topics to be covered

- Principle of eigenstructure assignment
- Low sensitivity eigenstructure assignment
- More study on eigenstructure assignment and Matlab toolboxes
- ROBUST EIGENSTRUCTURE ASSIGNMENT: NEW APPROACH
- Eigenstructure Assignment Applied to the Longitudinal Control of Aircraft
- Project
- References
Principle of eigenstructure assignment

An eigenvalue and eigenvector of a square matrix $A$ are, respectively, a scalar $\lambda$ and a nonzero vector $v$ that satisfy

$$Av = \lambda v$$

What is eigenstructure???????
What is eigenstructure assignment

- Eigenstructure assignment is a design technic which may be used to design the entire eigenstructure (eigenvalues and right or left eigenvectors) of a closed-loop linear system via a constant gain full state or output feedback control law.

It consists, essentially, of the following steps:

- 1) Choose a set of possible closed-loop Eigenvalues (or poles).
- 2) Compute the associated so-called allowable eigenvector subspace, which describe the freedom available for closed-loop eigenvector assignment.
- 3) Select specific eigenvectors from the allowable eigenvector subspaces according to some design strategies.
- 4) Calculate a control law, appropriate to the chosen eigenstructure.
State feedback and output feedback

State feedback

Output feedback
Definition of eigenstructure assignment

consider the following linear MIMO time invariant state space control system:

\[ \begin{align*}
\delta x &= Ax + Bu \\
y &= Cx 
\end{align*} \]

A, B, C are system matrices

\[ \begin{align*}
x &\in \mathbb{R}^n \\
u &\in \mathbb{R}^m \\
y &\in \mathbb{R}^r 
\end{align*} \]

the state vector

the control input vector

the output vector

Without loss of generality the following assumption is made for the system:

\[ \text{rank}(B) = m \text{ and } \text{rank}(C) = r \]

If either of these conditions are violated then:
Definition of eigenstructure assignment

If we apply the linear output feedback control law to the open-loop system

\[ u = Ky \]

If either of these conditions are violated then:

\[ \delta x = (A + BKC)x \]

Now, let us define a closed loop self conjugate eigenvalue set

\[ \Lambda = \{ \lambda_i : \lambda_i \in \mathcal{C}, i = 1, 2, ..., \tilde{n} \} \]

For an uncontrollable and/or unobservable system, the uncontrollable and/or unobservable open-loop eigenvalues should be included in the

\[ \Lambda = \{ \lambda_i : \lambda_i \in \mathcal{C}, i = 1, 2, ..., \tilde{n} \} \]

Output feedback eigenstructure cannot change those eigenvalues, but their corresponding eigenvectors may properly be chosen to improve the insensitivity of the closed-loop matrix and the robustness of the closed-loop system.
Right and left eigenvectors (and generalized eigenvectors)

For single closed-loop eigenvalues

\[(\lambda_i I - A - BKC)R_i = 0\]
\[L_i^T(\lambda_i I - A - BKC) = 0\]

\[R = [R_1, R_2, \ldots, R_{\tilde{n}}] \in \mathbb{C}^{n \times n}\]

spectral matrix

\[L = [L_1, L_2, \ldots, L_{\tilde{n}}] \in \mathbb{C}^{n \times n}\]

For multiple closed-loop eigenvalues, in general, we have the following form

\[\sum_{j=1}^{r_i} p_{ij} = q_i, \quad \sum_{i=1}^{\tilde{n}} q_i = n\]
Right and eigenvectors

\[ D_λ = \begin{bmatrix} D_1 & & \\ & D_2 & \\ & & \ddots \\ & & & D_n \end{bmatrix} \in \mathbb{C}^{n \times n} \]

\[ D_i = \begin{bmatrix} D_{i1} & & \\ & D_{i2} & \\ & & \ddots \\ & & & D_{in} \end{bmatrix} \in \mathbb{C}^{q_i \times q_i} \]

\[ D_{ij} = \begin{bmatrix} \lambda_i & 1 & & \\ & \lambda_i & 1 & \\ & & \ddots & \lambda_i \\ & & & 1 \end{bmatrix} \in \mathbb{C}^{p_{ij} \times p_{ij}} \]

\[ (λ_i I - A - BKC) R_{ij,k} = -R_{ij,k-1}, \quad R_{ij,0} = 0 \]

\[ R = [R_1, R_2, ..., R_n] \in \mathbb{C}^{n \times n} \]
\[ R_i = [R_{i1}, R_{i2}, ..., R_{in}] \in \mathbb{C}^{n \times q_i} \]
\[ R_{ij} = [R_{ij,1}, R_{ij,2}, ..., R_{ij,p_{ij}}] \in \mathbb{C}^{n \times p_{ij}} \]
Similarly for left eigenvectors, we have

\[ L_{i,j,k}^T (\lambda_i I - A - BKC) = -L_{i,j,k-1}^T, \quad L_{i,j,0} = 0 \]

\[
L = [L_1, L_2, \ldots, L_{\tilde{n}}] \in \mathbb{C}^{n \times n} \\
L_i = [L_{i,1}, L_{i,2}, \ldots, L_{i,r_i}] \in \mathbb{C}^{n \times q_i} \\
L_{ij} = [L_{ij,1}, L_{ij,2}, \ldots, L_{ij,p_{ij}}] \in \mathbb{C}^{n \times p_{ij}}
\]

By theorem, we have

\[ R^{-1} = L^T \]
Role of the system eigenstructure

Now we know what eigenstructure is, so we outline the significance of the eigenstructure in terms of the system time response. For the sake of simplicity, it is assumed that all eigenvalues are real and distinct, so we have the following spectral and modal matrices:

\[ D_{\Lambda} = \text{diagonal}(\lambda_i) \]
\[ R = [R_1 R_2 \ldots R_n] \]
\[ L = [l_1 l_2 \ldots l_n] \]

(in general case \( A \) is of a Jordan form)

to yield a solution to the state equations a transformation is performed so that the closed-loop matrix \( A+BKC \) takes on the diagonal form. Let

\[ x = Rz \]

where the vector \( z \) is a new variable vector. Applying this transformation to the system gives

\[ \dot{z} = R^{-1}(A+BKC)Rz \]
\[ y = CRz \]
The role of the system eigenstructure

It is clear from the relationship between the eigenvalues and eigenvectors that the solution is

\[ y = CRe^D \Lambda^t R^{-1} x(0) = CRe^D \Lambda^t L^T x(0) \]

and this may be written as

\[ y = \sum_{i=1}^{n} CR_i e^{\lambda_i t} L_{i}^T x(0) \]

the eigenstructure plays a key role in the response of the system. It can be seen that the transient response of the system is characterized by eigenvalues together with the right and left eigenvectors.
The role of the system eigenstructure

The eigenvalues determine the decay (or growth) rate of the response.
the right eigenvectors fix the shape of the response.
the product of initial condition $x(0)$ and left eigenvectors determines the amount each mode is excited in the response.
a judicious choice of a left eigenvector could prevent a mode from being excited by a known structure for the initial condition vector by choosing $L_i$ such that $L_i^T \ast x(0) = 0$

the role of system eigenstructure assignment in the forced response of the system:

$$u = K(y - r)$$

where $r$ is the reference input or desired output.
The role of the system eigenstructure

The role of the system eigenstructure

The forced response of the system is

\[ y(t) = CRe^{D_{st}}L^T x(0) + CR \int_0^t e^{D_{st}-\tau}L^T Br(\tau) d\tau \]

The product \( L^T * B \) indicates how much a particular input excites certain modes.

It may be important that a certain input has little (or ideally no) effect on specific modes of the system (modal decoupling). Therefore, in order to provide effective shaping of the response of a system, both left and right eigenvector assignment, in addition to eigenvalue assignment (pole placement), must certainly be considered together.
Allowable eigenvector subspace

This subspace consists of a function of the system input, output and state matrices and the choice of closed-loop eigenvalues. How to calculate the subspace?

We have two cases:

Case 1: The sets of open and closed-loop eigenvalues have no elements in common

$$|\lambda I - A| \neq 0$$

Case 2: The sets of open and closed-loop eigenvector intersect

For example, see the uncontrollable open-loop eigenvalues must be in the closed-loop eigenvalue set. We have different methods for real eigenvectors and complex eigenvectors.
Allowable eigenvector subspace

Case 1:
The $i$-th allowable right eigenvector may be chosen from a linear combination of the columns of

$$P_{R,i} = (\lambda_i I - A)^{-1}B$$

And the $i$-th allowable left eigenvector may be chosen as a linear combination of the columns of

$$P_{L,i} = (\lambda_i I - A^T)^{-1}C^T$$
Allowable eigenvector subspace

The dimension of each subspace is given by

\[
\dim(P_{R,i}) = m
\]

\[
\dim(P_{L,i}) = r
\]

The later two equations show that an allowable i-th right (left) closed-loop eigen vector may be chosen with m (r) degrees of freedom, i.e., that the number of entries of a right (left) eigenvector that may be exactly assigned is m (r).
Allowable eigenvector subspace

• For the case of the calculation of allowable eigenvector subspaces corresponding to complex eigenvalues, we define

\[ \lambda_i = \lambda_i^{re} + j\lambda_i^{im} \]
\[ R_i = R_i^{re} + jR_i^{im} \]

• The real and imaginary parts of of the allowable right eigenvector corresponding to the i-th complex eigenvalue may be chosen from a linear combination of the columns of

\[ P_{R,i}^c = \tilde{A}^{-1}\tilde{B} \]

• Where

\[ \tilde{A} = \begin{bmatrix} \lambda_i^{re}I - A & -\lambda_i^{im}I \\ \lambda_i^{im}I & \lambda_i^{re}I - A \end{bmatrix} \]
\[ \tilde{B} = \begin{bmatrix} B & 0 \\ 0 & B \end{bmatrix} \]
Allowable eigenvector subspace

Case 2:
For assignment right eigenvector of real eigenvalues, we should have

$$\begin{bmatrix} A - \lambda_i I & B \end{bmatrix} \begin{bmatrix} R_i \\ W_i \end{bmatrix} = 0$$

Which:

$$W_i = KCR_i$$
Allowable eigenvector subspace

Case 2:

For assignment right eigenvector of real eigenvalues, we should have

\[
\begin{bmatrix}
L_i^T & V_i^T
\end{bmatrix}
\begin{bmatrix}
A - \lambda_i I \\
C
\end{bmatrix} = 0
\]

Which:

\[
V_i = K^T B^T L_i
\]
Allowable eigenvector subspace

- And for assignment of right eigenvectors associated with complex eigenvalues we should satisfy the following equation:

\[
\begin{bmatrix}
\lambda_i^{re} I - A & -\lambda_i^{im} I & B & 0 \\
\lambda_i^{im} I & \lambda_i^{re} I - A & 0 & B
\end{bmatrix}
\begin{bmatrix}
R_i^{re} \\
R_i^{im} \\
W_i^{re} \\
W_i^{im}
\end{bmatrix} = 0
\]

- There is a similar equation for left eigenvectors.

- These problems may be calculated by performing the singular value decomposition (SVD) or orthogonal triangular decomposition (OTD) method.
Low sensitivity eigenstructure assignment
Low sensitivity Eigen structure assignment

- Correct response shaping is not the only aim

- There are often exist perturbation or parameter variation in system

- Three different measures for eigenvalue sensitivity:
  - Individual eigenvalue sensitivity
  - Overall vector sensitivity
  - Lower bound of overall conditioning
Individual Eigenvalue sensitivity

- It is shown that Sensitivity of $i$-th eigenvalue of a matrix to perturbation in some or all of its elements may be given by the expression:

$$\eta_i(R, L) = \frac{||L_i||_2||R_i||_2}{|L_i^T R_i|}$$

- $R_i$ and $L_i$ are the right and left eigenvectors of the matrix $X$ and

$$\eta_i(R, L) \geq 1$$

Individual Eigenvalue sensitivity

Example:

• Given the right eigenvector matrix $R$

\[
R = \begin{bmatrix}
5 & 1 & 2 \\
3 & 7 & 4 \\
0 & 2 & 6 \\
\end{bmatrix}
\]

• The left eigenvector is: $L = R^{-T}$

• Individual Eigenvalue sensitivity of the matrix are:

\[
\eta_1(R, L) = 1.2621 \\
\eta_2(R, L) = 1.6885 \\
\eta_3(R, L) = 1.5541
\]
Individual Eigenvalue sensitivity

- If perturbation \( o(\varepsilon) \) accrue then

\[
\tilde{\lambda}_i = \lambda_i + o(n\eta_i \varepsilon)
\]

- An eigenvalue is said to be perfectly conditioned if \( \eta_i \) is equal to unity it gives smallest change in eigenvalue position

Overall vector sensitivity

- The overall eigenvalue sensitivity of the matrix is defined as:

\[ \eta(R) = \| R \|_2 \| R^{-1} \|_2 \]

- Example:

\[ R = \begin{bmatrix} 5 & 1 & 2 \\ 3 & 7 & 4 \\ 0 & 2 & 6 \end{bmatrix} \]

- The overall sensitivity is:

\[ \eta(R) = 3.0795 \]
Upper bound of eigenvalue sensitivities

- Upper bound on individual eigenvalue sensitivity:

\[
\max_i \eta_i(R, L) \leq \eta(R)
\]

- In Example:

\[
\eta_1(R, L) = 1.2621 \\
\eta_2(R, L) = 1.6885 \\
\eta_3(R, L) = 1.5541
\]

\[
\eta(R) = 3.0795
\]

More study on eigenstructure assignment and Matlab toolboxes
Eigenstructure assignment toolbox

Matlab-EAT toolbox has been written to enable control system designers to have:

- Flexibility in the choice of design strategy
- Providing an opportunity for a tutorial introduction to eigenstructure assignment
- Functions enable the designer to generate stable and minimally sensitive or robust control for both state and output feedback multivariable control
EAT functions

- Basic eigenstructure assignment
- Recursive eigenstructure assignment
- Multi objective robust eigenstructure assignment
- Eigen structure assignment for sampled-date systems
- Eigenstructure assignment for dynamical compensator
- Fundamental functions related to eigenstructure assignment

Since an allowable closed-loop eigenvector belongs to a subspace of dimension which is generally less than the number of the states \( \Rightarrow \) we can not assign any other closed loop eigenvector except those that belong to the span of the corresponding allowable subspace.

Choosing the eigenvector from allowable eigenvector subspace which minimizes least square Error defined by:

\[
\min_{\vec{R}_i} \| \vec{R}_i - \vec{R}_{di} \|_2
\]

\( \vec{R}_{di} \) is the desired closed-loop eigenvector

\( \vec{R}_i \) is an achievable eigenvector
## Basic eigenstructure assignment

- **Non-iterative (Direct)**

<table>
<thead>
<tr>
<th>Basic eigenstructure assignment</th>
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</thead>
<tbody>
<tr>
<td>projea</td>
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<td>direa</td>
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</tbody>
</table>

- **Projea**: projects the desired eigenvector into allowable eigenvector subspace using the least square method for a linear state feedback system
Recursive Eigenstructure Assignment

- Considering the overall measure of conditioning of the eigen problem.
- The sensitivity of the closed-loop system matrix is defined as:
  \[ \eta = \| R \|_2 \| L \|_2 \]
- R and L are the right and left achieved eigenvector matrices.
- It chooses recursively the closed-loop eigenvector that the whole sensitivity of the eigenvector matrix is minimized.
- Recursive state feedback eigenstructure assignment & recursive output feedback eigenstructure assignment
Multi objective robust eigenstructure assignment

- There are often a number of design objectives.
- They are conflicting, no design exists to be considered best with respect to all objectives.
- There is an inevitable tradeoff between design objectives.
- Scalar summation of all weighted objectives in one cost function means it is not clear how each objective is affected by the controller. Solution: using individual cost functions.
Using individual cost functions

\[ \phi_i(K) = \frac{\| R_i \|_2 \| L_i \|_2}{|R_i^T L_i|}, \quad \text{for} \quad i = 1, 2, \ldots, n \]
\[ \phi_{n+1}(K) = \| S \|_{\infty, \mathcal{F}} \]
\[ \phi_{n+2}(K) = \| KS \|_{\infty, \mathcal{F}} \]
\[ \phi_{n+3}(K) = \| GKS \|_{\infty, \mathcal{F}} \]

- It is not possible to minimize all by a controller so we formulate a multi objective controller such that:
- Also minimize:

\[ \phi_i(K) \leq \varepsilon_i, \quad \text{for} \quad i = 1, 2, \ldots, n + 3 \]

\[ J_F = \sum_{i=1}^{n+3} \frac{\phi_i(K)}{\varepsilon_i} \]

- The function mobjea performs multi objective eigenstructure assignment which uses genetic algorithm methods
ROBUST EIGENSTRUCTURE ASSIGNMENT: NEW APPROACH

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ROBUST EIGENSTRUCTURE ASSIGNMENT: NEW APPROACH

• Let $Q^H AQ = D + N$ be Schur decomposition of $A$

• and $N$ is strictly upper triangular matrix and $Q$ is an appropriate unitary matrix

$$D = \text{diag}(\lambda_1, \ldots, \lambda_n)$$

• When the closed-loop system matrix in $\dot{x}(t) = (A + BKC)x(t)$ is faced with a perturbation matrix $E$, if $K$ is designed such that $N$ is equal to zero, the upper bound on variations of closed-loop eigenvalues will be minimized ($A+BKC$ will be low sensitive). $N$ is the corresponding matrix, obtained from Schur decomposition of the closed loop system matrix.
Matrix $A$ is normal if and only if the matrix $N$ is equal to zero in the Schur decomposition of matrix $A$.

$A$ is normal:

$\|A\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A)|^2$

$A+BKC$ is normal:

$\|A+BKC\|_F^2 = \sum_{i=1}^{n} |\lambda_i(A+BKC)|^2$
Algorithm:

- Select the set of desired closed-loop eigenvalues, according to the stability and dynamic response characteristic requirements of the system.
- Minimize the following cost function, in order to solve the robust eigenstructure assignment problem:

\[ J = \|A + BKC\|_F^2 - \sum_{i=1}^{n} \lambda_i (A + BKC)^2 \]
ROBUST EIGENSTRUCTURE
ASSIGNMENT: NEW APPROACH

Example:

\[
A = \begin{bmatrix}
-0.1094 & 0.0628 & 0 & 0 & 0 \\
1.3060 & -2.1320 & 0.9807 & 0 & 0 \\
0 & 1.5950 & -3.1490 & 1.5470 & 0 \\
0 & 0.0355 & 2.6320 & -4.2570 & 1.8550 \\
0 & 0.0023 & 0 & 0.1636 & -0.1625 \\
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 & 0.0638 & 0.0838 & 0.1004 & 0.0063 \\
0 & 0 & -0.1396 & -0.2060 & -0.0128 \\
\end{bmatrix}^T
\]

\[
C = I_{5 \times 5}
\]

\[
K_1 = \begin{bmatrix}
-150.7102 & 27.9662 & -46.1743 & 54.7426 & 56.7101 \\
-58.3445 & 16.8117 & -4.7370 & 20.5022 & 70.0154 \\
\end{bmatrix}
\]

\[
K_2 = \begin{bmatrix}
-245.1060 & -2.6951 & -35.4448 & 31.6681 & 66.4664 \\
-131.4648 & -10.8048 & 9.3406 & -5.2734 & 72.5363 \\
\end{bmatrix}
\]

This method

Other robust method (spectral condition)

Not robust k

\[
K_3 = 10^5 \times \begin{bmatrix}
-1.6734 & -0.0798 & 0.0267 & -0.0434 & 0.2376 \\
-4.3434 & -0.2075 & 0.0706 & -0.1139 & 0.6171 \\
\end{bmatrix}
\]
Results: The desired eigenvalues of the closed-loop system are chosen to be: \{-0.5, -1, -2, -3, -4\}.

<table>
<thead>
<tr>
<th>Changes in all elements of the matrix $A$</th>
<th>Feedback matrix</th>
<th>closed-loop eigenvalues</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$K_1$</td>
<td>-0.5003, -1.0003,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1.9998, -2.9995,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-4.0098</td>
</tr>
<tr>
<td>+1%</td>
<td>$K_2$</td>
<td>-0.5004, -1.0000,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-1.9987, -3.0038,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-4.0069</td>
</tr>
<tr>
<td></td>
<td>$K_3$</td>
<td>-0.2863, -0.9473,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-2.9514 ± 1.6005i,</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-3.3733</td>
</tr>
</tbody>
</table>
Results: The desired eigenvalues of the closed-loop system are chosen to be: \{-0.5, -1, -2, -3, -4\}.

<table>
<thead>
<tr>
<th></th>
<th>(K_1)</th>
<th>(-0.4997, -0.9997,)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>(-2.0002, -3.0005,)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.9901)</td>
</tr>
<tr>
<td></td>
<td>(K_2)</td>
<td>(-0.4996, -1.0000,)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-2.0013, -2.9962,)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.9931)</td>
</tr>
<tr>
<td></td>
<td>(K_3)</td>
<td>(-0.8783,)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-0.7157 \pm 0.6801i)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(-3.2207, -4.9597)</td>
</tr>
</tbody>
</table>
Results: The desired eigenvalues of the closed-loop system are chosen to be: \{-0.5, -1, -2, -3, -4\}.

<table>
<thead>
<tr>
<th>(K_1)</th>
<th>-0.5028, -1.0035, -1.9980, -2.9979, -4.0959</th>
</tr>
</thead>
<tbody>
<tr>
<td>(K_2)</td>
<td>-0.5038, -0.9998, -1.9878, -3.0376, -4.0691</td>
</tr>
<tr>
<td>(K_3)</td>
<td>0.0437, -0.9375, -3.1911 ± 5.3831i, -3.3221</td>
</tr>
</tbody>
</table>

+10%
Results: The desired eigenvalues of the closed-loop system are chosen to be: \{-0.5, -1, -2, -3, -4\}.

<table>
<thead>
<tr>
<th></th>
<th>$K_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>-10%</td>
<td>{-0.4971, -0.9965, -2.0016, -3.0084, -3.8982}</td>
</tr>
<tr>
<td></td>
<td>$K_2$</td>
</tr>
<tr>
<td>-10%</td>
<td>{-0.4962, -1.0002, -2.0141, -2.9608, -3.9306}</td>
</tr>
<tr>
<td></td>
<td>$K_3$</td>
</tr>
<tr>
<td></td>
<td>{1.4827, 0.7596, -0.9132, -3.2422, -8.4888}</td>
</tr>
</tbody>
</table>
Aruna N. S.
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Scientist Engineer “SE”-CLD, CGDG, V S S C- Thiruvananthapuram
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Professor Department of Electrical Engineering-College of Engineering Trivandrum
It is well known that when an aircraft performs some angular maneuver a coupled response is obtained, that is, if the vehicle performs a roll and yaw maneuver this will also cause a response in pitch oscillation. This behavior is undesirable from the point of view of vehicle performance.
Introduction

- Eigenvalues are chosen to obtain desired damping and rise time.
- Eigenvectors are assigned such that we decouple the plant.
- Feedforward gain is computed which ensure steady-state tracking of the pilot’s command.
- The application of Eigenstructure Assignment to conventional flight control design as been described by Shapiro [1]. A design methodology that uses Eigenstructure Assignment to obtain decoupled aircraft motions has been described by Sobel [2]. Study of Eigenstructure Assignment on a MIMO system was done from [3], [4], [5]. In this article, we use Eigenstructure Assignment to design multimode flight control system [6] and [7]. The longitudinal design is illustrated by using the unstable dynamics of an advanced fighter aircraft [8] and [9]. In this paper controller is designed using Eigenstructure Assignment for tracking both angle of attack and flight path angle. The study of Toolbox for Eigenstructure Assignment [10] was done and applied for the the work.
Introduction

Nomenclature

$\alpha = \text{angle of attack}$
$q = \text{pitch rate}$
$\theta = \text{pitch angle}$
$\gamma = \text{flight path angle}$
$\delta_e = \text{elevator deflection}$
$\delta_f = \text{flaperon deflection}$
$\delta_{ec} = \text{elevator deflection command}$
$\delta_{fc} = \text{flaperon deflection command}$
$\theta_c = \text{pilot's pitch attitude command}$
$\gamma_c = \text{pilot's flight path angle command}$
$h = \text{altitude rate}$

$u = [\delta_{ec} \ \delta_{fc}]^T$

$x = \begin{bmatrix}
\alpha \\
q \\
\gamma \\
\delta_e \\
\delta_f
\end{bmatrix}$

$y = [q \ \ n_{zp} \ \ \gamma \ \ \delta_e \ \ \delta_f]^T$
Introduction

- Consider an aircraft modeled by the linear time-invariant matrix differential equation given by

\[
\begin{align*}
x &= Ax + Bu \\
y &= Cx + Du
\end{align*}
\]

It is assumed that the system is controlled by a static output feedback and that an input reference is shaped by a feed forward gain $H$. The control law will be given by

\[
u = Ky + Hz_R
\]

Where $z$ stands for the reference input. If $D=0$:

\[
\dot{x} = (A + BKC)x + BHz_R
\]
It is assumed that the system is controlled by a static output feedback and that an input reference is shaped by a feed forward gain $H$. 
Plant specifications

- The eigenvalue of the open-loop system from matrix A are given by:
  - \( \lambda_1 = 5.452 \)
  - \( \lambda_2 = -7.662 \)
  - \( \lambda_3 = 0 \)
  - \( \lambda_4 = -20 \)
  - \( \lambda_5 = -20 \)

  pitch attitude mode elevator actuator mode flap eron actuator mode

\[
A = \begin{bmatrix}
-1.341 & 0.9933 & 0 & -0.1689 & -0.2518 \\
43.223 & -0.8693 & 0 & -17.251 & -1.5766 \\
1.341 & 0.0067 & 0 & 0.1689 & 0.2518 \\
0 & 0 & 0 & -20 & 0 \\
0 & 0 & 0 & 0 & -20
\end{bmatrix}
\]

\[
B = \begin{bmatrix}
0 \\
0 \\
0 \\
0 \\
20
\end{bmatrix}
\]

\[
x = \begin{bmatrix}
\alpha \\
q \\
\gamma \\
\delta_e \\
\delta_f
\end{bmatrix}
\]

\[
y = [q \ n_{zp} \ \gamma \ \delta_e \ \delta_f]^T
\]
Plant specifications

\[ u = Ky + Hz_R \]

\[ x = (A + BKC)x + BHx_R \]

The feedback gain matrix $K$ will exactly assign $r$ eigenvalues. It will also assign the corresponding eigenvectors, provided that they were chosen to be in the subspace spanned by the columns of

\[ N_i = (\lambda_i I - A)^{-1}B \]

Since there are five outputs and two control inputs, we may place all the closed loop poles as well as assign the eigenvectors within two-dimensional subspaces. This roughly corresponds to selecting two components of each eigenvector arbitrarily.
Desirable and achievable eigenvectors

Desirable eigenvectors

\[
\begin{bmatrix}
1 \\
-1 \\
0 \\
x \\
x
\end{bmatrix} + j
\begin{bmatrix}
-1 \\
1 \\
0 \\
x \\
x
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
x \\
x \\
x
\end{bmatrix}
\begin{bmatrix}
\alpha \\
q \\
\gamma \\
\delta_e \\
\delta_f
\end{bmatrix}
\]

Achievable eigenvectors

\[
\begin{bmatrix}
-0.93 \\
1 \\
0 \\
x
\end{bmatrix} + j
\begin{bmatrix}
-9.5 \\
0 \\
0 \\
x
\end{bmatrix}
\begin{bmatrix}
x \\
x \\
x \\
x
\end{bmatrix}
\begin{bmatrix}
-0.051 \\
1.07 \\
-0.006 \\
1
\end{bmatrix}
\begin{bmatrix}
\alpha \\
q \\
\gamma \\
\delta_e \\
\delta_f
\end{bmatrix}
\]
### Final design

#### Pitch Pointing Control Law

<table>
<thead>
<tr>
<th>Desired Eigenvalues</th>
<th>Feedforward Gain $H$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\lambda_{1,2} = -10.6 \pm 4.2i$</td>
<td>$\begin{bmatrix} 32.51 &amp; 29.76 \ 37.52 &amp; -46.59 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\lambda_3 = -1$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_4 = -18$</td>
<td></td>
</tr>
<tr>
<td>$\lambda_5 = -18.5$</td>
<td></td>
</tr>
</tbody>
</table>

#### Feedback Gain, $K$

$$
\begin{bmatrix}
q & n & zp \\
-1.18 & -0.17 & -29.75 & -0.24 & 0.36 \\
1.05 & 0.21 & 46.59 & 0.65 & -0.67
\end{bmatrix}
$$
Time responses of output when step command is given to elevator deflection command
Time responses of output when step command is given to Flaperon deflection command
Project

Consider the following system

\[ A = \begin{bmatrix} -10 & 3 & 2 \\ 0 & 5 & 1 \\ 2 & 3 & -8 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \quad C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \]

- Find gain(k) such that
  i) The system becomes stable
  ii) Try to make A+Bk Low sensitive (Robust) to additive perturbation.
- Plot step response for both systems with state feedback and without feedback (Using Matlab simulink is suggested).
- Perturb system matrix (A+Bk) with some additive perturbation matrix E show that the feedback system matrix is not really sensitive to the perturbation (E is an arbitrary matrix).
- Try to make the steady state error zero.
References

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- Liu G.P and R.J. Patton Eigenstructure assignment toolbox for use with Matlab Department of electronic Engineering University of Hull, UK
Thanks For your attention