In The Name Of GOD

Input-output pairing

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Topics to be covered include:

- Decentralized control
- RGA
  - DRGA
  - ROmA
  - REGA
  - RNGA
- GERAMIAN
- RGA Analysis of Uncertain Multivariable Plants
- ARG A
Diagonal controller (decentralized control)

Another simple approach to multivariable controller design is to use a diagonal or block diagonal controller $K(s)$. This is often referred to as decentralized control.

Despite the availability of sophisticated methods for designing multivariable control systems, decentralized control remains dominant in industry applications mainly due to: (1) it requires fewer parameters to tune which are easier to be understood and implemented; and (2) loop failure tolerance of the resulting control system can be assured during the design phase. Therefore, they are more often used in process control applications. The design of decentralized control systems involves two steps:

1. The choice of pairings (control configuration selection)
2. The design (tuning) of each controller $k_i(s)$
Diagonal controller (decentralized control)

the primary task in the design of decentralized control systems is to determine loop configuration, i.e. pair the manipulated variables and controlled variables to achieve the minimum interactions among control loops so that the resulting multivariable control system mostly resembles its single-input single-output counterparts and the subsequent controller tuning is largely facilitated by SISO design techniques.

There are several methods for input output pairing, such as:
1- RGA
2- Balanced realization
3- Geramian Matrix
4- Hankel Norms

For each above methods we have several procedures.
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RGA

RGA (relative gain array)
The RGA (Bristol, 1966) of a non-singular square complex matrix $A$ is square complex matrix defined as:

$$RGA(A) = \Lambda(A) = A \times (A^{-1})^T$$

The RGA for a non-square complex matrix $A$ is:

$$RGA(A) = \Lambda(A) = A \times (A^\dagger)^T$$

We show that the RGA provides a measure of interactions. Let $u_i$ and $y_i$ denote a particular input-output pair for the multivariable plant $G(s)$, and assume that our task is to use $u_i$ to control $y_i$. Bristol argued that there will be two extreme cases:

- All other loops open: $u_k = 0$, for all $k \neq j$
- All other loops closed with perfect control: $y_k = 0$, for all $k \neq i$

Perfect control is only possible at steady-state, but it is a good approximation at frequencies within the bandwidth of each loop. We now evaluate our gain $\frac{\partial y_i}{\partial u_j}$ for the 2 extreme cases:

$$g_{ij} = [G]_{ij} \text{ is the } ij^{th} \text{ element of } G$$

$$\hat{g}_{ij} \text{ is the inverse of the } ji^{th} \text{ element of } G^{-1}$$
RGA

We note that

\[ y = Gu \Rightarrow \left( \frac{\partial y_i}{\partial u_j} \right)_{u_k=0, k \neq j} = [G]_{ij} \quad \text{and} \quad u = G^{-1}y \Rightarrow \left( \frac{\partial u_j}{\partial y_i} \right)_{y_k=0, k \neq i} = [G^{-1}]_{ji} \]

Bristol argued that the ratio between the above gains is a useful measure of interaction, and defined the ij’th relative gain as:

\[ \lambda_{ij} \triangleq \frac{g_{ij}}{g_{ij}} = [G]_{ij}[G^{-1}]_{ji} \]

In this section we provide two useful rules for pairing inputs and outputs.
1. To avoid instability caused by interactions in the crossover region one should prefer pairings for which the RGA matrix in this frequency range is close to identity.
2. To avoid instability caused by interactions at low frequencies one should avoid pairings with negative steady state RGA elements.

Properties:
1. It is independent of input and output scaling. 2. Its rows and columns sum to 1
3. The RGA is the identity matrix if G is upper or lower triangular.
4. Large RGA element at frequencies important for control indicate that the plant is fundamentally difficult to control due to strong interaction and sensitivity to uncertainty.
Example:

\[ G(0) = \begin{bmatrix} 10.2 & 5.6 & 1.4 \\ 15.5 & -8.4 & -0.7 \\ 18.1 & 0.4 & 1.8 \end{bmatrix} \quad \Lambda(0) = \begin{bmatrix} 0.96 & 1.45 & -1.41 \\ 0.94 & -0.37 & 0.43 \\ -0.9 & -0.07 & 1.98 \end{bmatrix} \]

Moreover, using steady-state gain alone may result in incorrect interaction measures and consequently loop pairing decisions, since no dynamic information of the process is taken into consideration.

Many improved approaches, RGA-like, have been proposed and described in all process control textbooks, for defining different measures of dynamic loop interactions. This methods are:

- DRGA (dynamic RGA)
- ROMA (relative omega array)
- REGA (relative effective gain array)
- RNGA (relative normalized gain array)
- ARGA (absolute relative gain array)
To overcome the limitations of RGA based loop pairing criterion, several pairing methods have later been proposed by using the dynamic Relative Gain Array (DRGA) to consider the effects of process dynamics, which employ the transfer function model instead of the steady state gain matrix to calculate RGA. In RGA, the denominator involved achieving perfect control at all frequencies, while the numerator was simply the open-loop transfer function.

Recently, McAvoy et al. (2003) proposed a significant DRGA approach. Using the available dynamic process model, a proportional output optimal controller is designed based on the state space approach and the resulting controller gain matrix is used to define a DRGA.

Several examples in which the normal RGA gives the inaccurate interaction measure and wrong pairings were studied and in all cases the new DRGA method gives more accurate interaction assessment and the best pairings. However, DRGA is often controller dependent, which makes it more difficult to calculate and to be understood by practical control engineers.
To combine the advantages of both RGA and DRGA, Xiong et al. introduced a relative effective gain array (REGA) based loop pairing criterion by employing the steady-state gain and bandwidth of the process transfer function element. Since the REGA considers both the steady-state and the transient information of the process, it provides a more comprehensive description for loop interactions. Another advantage of REGA is that it is controller independent which is more superior to other existing loop pairing methods. However, since the calculation of REGA depends on the critical frequency point of individual element, different selection criteria for critical frequency points result in different REGAs, subsequently, cause uncertainties in control structure configurations.

Example:

\[
G(s) = \begin{pmatrix}
\frac{5e^{-s}}{100s+1} & \frac{e^{-4s}}{10s+1} \\
-\frac{5e^{-4s}}{10s+1} & \frac{5e^{-s}}{100s+1}
\end{pmatrix}
\begin{pmatrix}
0.9840 & 0.0160 \\
0.0160 & 0.9840
\end{pmatrix}
\begin{pmatrix}
0.0476 & 0.9524 \\
0.9524 & 0.0476
\end{pmatrix}
\]

Diagonal pairing \(\omega_{c,ij} = \omega_{u,ij}\) with \(\arg(\tilde{g}_{ij}(j\omega_{u,ij})) = -\pi\)

Off-diagonal pairing \(\omega_{c,ij} = \omega_{b,ij}\) with \(|\tilde{g}_{ij}(j\omega_{b,ij})| = \sqrt{2}/2\)

REGA is critical frequency dependent, which means, with selecting different critical frequencies, REGA suggests different control structure configurations.
ROmA

The logic behind Relative Omega Array (ROmA) index was to measure interactions in MIMO systems, capturing information from critical frequencies variation in the passage from open loop to closed loop.

Example 1:

\[
G(s) = \begin{bmatrix}
12.8e^{-s} & -18.9e^{-3s} \\
16.7s+1 & 21s+1 \\
6.6e^{-7s} & -19.4e^{-3s} \\
10.9s+1 & 14.4s+1
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
2.0094 & -1.0094 \\
-1.0094 & 2.0094
\end{bmatrix}
\]

\[
\Psi = F \otimes F^{-T} = \begin{bmatrix}
1.1133 & -0.1133 \\
-0.1133 & 1.1133
\end{bmatrix} = \text{ROmA}
\]

Pairings suggested by both methods is diagonal pairing \((y_1-u_1, y_2-u_2)\)

Example 2:

\[
G(s) = \begin{bmatrix}
e^{-s} & 1 \\
1+s & 1+s \\
-1 & e^{-2s} \\
1+s & 1+s
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}
\]

\[
\Psi = F \otimes F^{-T} = \begin{bmatrix}
-0.0461 & 1.0461 \\
1.0461 & -0.0461
\end{bmatrix} = \text{ROmA}
\]

By comparing this matrix with the ROmA matrix it may be observed that the suggested pairings is the off-diagonal one: \(y_1-u_2, y_2-u_1\). This result is in good agreement, in a critical case where the steady-state RGA does not suggest any preferential pairing.
a new method for interaction measurement. Through investigating both the steady-state and transient information of the process transfer function, the normalized gain is defined to provide a more comprehensive description of each process input to output channel. The relative normalized gain array (RNGA) is then introduced for loop interaction measurements.

The main advantages of this method are:

(1) Compared with RGA method, it considers not only the process steady-state information but also transient information.

(2) Compared with DRGA method, it also provides a comprehensive description of dynamic interaction among individual loops without requiring the specification of the controller type and with much less computation.

(3) Compared with REGA method, it requires even less calculation but resulting in an unique and optimal loop pairing decision.

(4) It is very simple for field engineers to understand and work out pairing decisions in practical applications.

Several examples, for which the RGA based loop pairing criterion gives an inaccurate interaction assessment, are employed to demonstrate the effectiveness of the proposed interaction measure and loop pairing criterion.
In this method for each $g_{ij}(s)$ element we have:

$$g_{ij}(s) = g_{ij}(j0) \times \bar{g}_{ij}(s)$$

As a accumulation of the difference between the expected and the real outputs of process $\hat{g}_{ij}(s)$; $A_{ij}$, in fact, is equal to the average residence time $\tau_{ar,ij}$ of $\hat{g}_{ij}(s)$, $\tau_{ar,ij} = \bar{A}_{ij}$.

Apparently, smaller $\tau_{ar,ij}$ indicates that the transfer function has fast response to input disturbance, while larger $\tau_{ar,ij}$ indicates the open-loop process has slower process dynamics. Therefore, the average residence time $\tau_{ar,ij}$ can effectively reflect the process dynamics of $\hat{g}_{ij}(s)$, and accordingly $g_{ij}(s)$.

$$\tau_{ar,ij} = \bar{A}_{ij} = \int_0^{\infty} [\bar{y}_i(\infty) - \bar{y}_i(t)] dt = \int_0^{\infty} [1 - (1 - e^{-(t-\theta_{ij})/\tau_{ij}})] dt$$

$$= \int_0^{\infty} e^{-(t-\theta_{ij})/\tau_{ij}} dt = \tau_{ij} + \theta_{ij}.$$
In this method, two important parameters for the process $g_{ij}(s)$ are obtained:

- Steady-state gain $g_{ij}(j0)$: the steady-state gain reflects the effect of the manipulated variable $u_j$ to the controlled variable $y_i$.

- Average residence time $\tau_{ij}$: the average residence time is accountable for the response speed of the controlled variable $y_i$ to manipulated variable $u_j$.

to use above both parameters for interaction measure and loop pairing, we now define the normalized gain (NG) $k_{N,ij}$ for a particular transfer function $g_{ij}(s)$ as:

$$k_{N,ij} = \frac{g_{ij}(j0)}{\tau_{ar,ij}}$$

indicates that a large value of $k_{N,ij}$ implies that the combination effect of the manipulated variable $u_j$ to the controlled variable $y_i$ and the response speed of the controlled variable $y_i$ to manipulated variable $u_j$ is large. Therefore, the loop pairing with large normalized gain $k_{N,ij}$ should be preferred. all elements of transfer function matrix $G(s)$, one can obtain the normalized gain matrix $K_N$ as:

$$K_N = [k_{N,ij}]_{n \times n} = G(j0) \odot T_{ar},$$

where $T_{ar} = [\tau_{ar,ij}]_{n \times n}$ and $\odot$ indicates element-by-element division.
Similar to the definition of relative gain, by replacing the steady-state gain matrix with the normalized gain matrix \( k_N \), we define the relative normalized gain (RNG) between output variable \( y_i \) and input variable \( u_j \), \( \phi_{ij} \), as the ratio of two normalized gains:

\[
\phi_{ij} = \frac{k_{N,ij}}{\hat{k}_{N,ij}}
\]

where \( k_{N,ij} \) is the effective gain between output variable \( y_i \) and input variable \( u_j \) when all other loops are closed. And relative normalized gain array (RNGA) is:

\[
\Phi = K_N \otimes K_N^{-T}
\]

The pairing rules is:
(i) the paired RGA elements are closest to 1.0
(ii) the NI is positive
(iii) all paired RGA elements are positive
(iv) large RGA elements should be avoided.
Example 1:

\[
G(s) = \begin{bmatrix}
\frac{12.8e^{-s}}{16.7s+1} & -\frac{18.9e^{-3s}}{21s+1} \\
\frac{6.6e^{-7s}}{10.9s+1} & -\frac{19.4e^{-3s}}{14.4s+1}
\end{bmatrix}
\]

\[
RGA = \begin{bmatrix}
\frac{2.0094}{-1.0094} & -1.0094 \\
2.0094 & \frac{1.5636}{-0.5636}
\end{bmatrix}
\]

\[
RNGA = \begin{bmatrix}
\frac{1.5636}{-0.5636} & -0.5636 \\
\frac{-0.5636}{1.5636}
\end{bmatrix}
\]

Example 2:

\[
G(s) = \begin{bmatrix}
\frac{5e^{-s}}{100s+1} & \frac{e^{-4s}}{10s+1} \\
-\frac{5e^{-4s}}{10s+1} & \frac{5e^{-s}}{100s+1}
\end{bmatrix}
\]

\[
RGA = \begin{bmatrix}
\frac{0.8333}{0.1667} & 0.1667 \\
0.1667 & \frac{0.8333}{0.1667}
\end{bmatrix}
\]

\[
RNGA = \begin{bmatrix}
\frac{0.0876}{0.9124} & 0.9124 \\
0.9124 & \frac{0.0876}{0.9124}
\end{bmatrix}
\]

RGA & RNGA have same result

RGA & RNGA have different result but which one is better?

decentralized controllers for both diagonal and off-diagonal pairings are designed respectively based on the IMC-PID controller. To evaluate the output control performance, we consider a unit step set-point change of all control loops one-by-one and the integral square error (ISE) of \(e_i(t) = r_i(t) - y_i(t)\) is used to evaluate the control performance

\[
ISE = \int_0^\infty e_i^2(t) \, dt
\]
The simulation results and ISE values are given in this figure. The results show that the off-diagonal pairing gives better overall control system performance.
Example 3:

$$G(s) = \begin{pmatrix}
\frac{e^{-9s}}{6s^2+17s+1} & \frac{-9e^{-5s}}{s^2+4s+1} & \frac{13e^{-3s}}{3s^2+35s+1} \\
\frac{-5e^{-13s}}{2s^2+19s+1} & \frac{8e^{-2s}}{s^2+33s+1} & \frac{7e^{-5s}}{s^2+3s+1} \\
\frac{-16e^{-3s}}{s^2+5s+1} & \frac{3e^{-7s}}{s^2+14s+1} & \frac{e^{-11s}}{3s^2+25s+1}
\end{pmatrix}$$

$$\Lambda(G(j0)) = \begin{pmatrix}
-0.0054 & 0.3981 & 0.6073 \\
-0.0992 & 0.6912 & 0.4080 \\
1.1046 & -0.0893 & -0.0153
\end{pmatrix}$$

$$T_{ar} = \begin{pmatrix}
26 & 9 & 38 \\
32 & 35 & 8 \\
8 & 21 & 36
\end{pmatrix},$$

$$K_N = \begin{pmatrix}
0.0385 & -1.0000 & 0.3421 \\
-0.1563 & 0.2286 & 0.8750 \\
-2.0000 & 0.1429 & 0.0278
\end{pmatrix}.$$
RNGA
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- RGA Analysis of Uncertain Multivariable Plants
- ARGA
Input-Output Pairing based on Cross Gramian Matrix

For the linear single-input, single-output, asymptotically stable, time invariant system $S(A, b, c)$ described by:

\[ y(t) = cx(t) \]
\[ \dot{x}(t) = Ax(t) + bu(t) \quad (1) \]

The controllability and observability gramian matrices are respectively defined as 2&3:

\[ W_c = \int_0^\infty e^{At}bb^T e^{A^Tt} dt \quad (2) \]
\[ W_o = \int_0^\infty e^{A^Tt}c^T ce^{At} dt \quad (3) \]

And these matrices can be computed by solving the linear matrix equations like 4 &5:

\[ W_c A^T + AW_c = -bb^T \quad (4) \]
\[ W_o A + A^T W_o = -c^T c \quad (5) \]
Gramian Matrix

Where, the system is controllable and observable if $W_c$ and $W_o$ are positive definite matrices.

Using the impulse response of the controllable and observable system, the cross-gramian matrix $W_{co}$ is defined as [6]:

$$ W_{co} = \int_{0}^{\infty} (e^{At}b)(e^{At}c^T)^T dt = \int_{0}^{\infty} e^{At}bce^{At} dt $$  \hspace{1cm} (6)

It is easily seen that the matrix $W_{co}$ can be computed by solving the following linear matrix equation:

$$ W_{co}A + AW_{co} = -bc $$ \hspace{1cm} (7)

Since the matrix $A$ is assumed to be stable, a unique solution existes. The matrix $W_{co}$ provides information about both controllability and observability and it is easily seen that the eigenvalues of the matrix $W_{co}$ are invariant under similarity transformation of the system.
Solving the Sylvester equation

Theorem 1 (distinct eigenvalues): consider the single input, single output, asymptotically stable, linear time invariant system, \( S(A_{n \times n}, B_{n \times 1}, C_{1 \times n}) \) where \( \lambda_n \) (i=1,…,n) are distinct eigenvalues of matrix \( A \) and \( v_n \) (i=1,…,n) are the corresponding eigenvectors, the cross Gramian matrix, \( W_{co} \) of this system can be computed as:

\[
W_{co} = -[I \ I \ ... \ I]\begin{bmatrix}
(A + \lambda_1 I)^{-1}bcv_1 & 0 & \cdots & 0 \\
0 & (A + \lambda_2 I)^{-1}bcv_2 & \cdots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \cdots & (A + \lambda_n I)^{-1}bcv_n
\end{bmatrix}[v_1 \ \ldots \ v_i \ \ldots \ v_n]^{-1}
\]

Example:

\[
\dot{x} = \begin{bmatrix}-4 & -2 & -0.5 \\1 & -1 & 0.5 \\4 & 4 & -1\end{bmatrix}x + \begin{bmatrix}0 \\0 \\1\end{bmatrix}u \quad y = [7 \ 3 \ 3]x
\]

Diagonal form of matrix \( A \) is:

\[
\Lambda = \begin{bmatrix}-1 & 0 & 0 \\
0 & -2 & 0 \\
0 & 0 & -3\end{bmatrix} \quad w_{co} = \begin{bmatrix}0.36 & 0.57 & 0.27 \\
-0.36 & -0.57 & -0.27 \\
0.91 & 1.25 & 0.58\end{bmatrix}
\]
Theorem 2 (repeated eigenvalues): consider the single input, single output asymptotically stable, linear time invariant system, $S(A_{n \times n}, B_{n \times 1}, C_{1 \times n})$ assume that $\lambda$ is a repeated eigenvalue of $A$ with multiplicity $n$. The Jordan form of $A$ is:

$$J = \begin{bmatrix}
\lambda & 1 & 0 & \cdots & 0 \\
0 & \lambda & 1 & \cdots & 0 \\
\vdots & 0 & \ddots & \ddots & \vdots \\
\vdots & \vdots & \ddots & \ddots & 1 \\
0 & 0 & 0 & 0 & \lambda
\end{bmatrix}$$

The cross-gramian matrix, of this system can be computed as:

$$W_{co} = \begin{bmatrix} I_n & I_n & \cdots & I_n \\
-(A+\lambda I)^{-1}b_{cv}(1) & (A+\lambda I)^{-2}b_{cv}(1) & \cdots & (A+\lambda I)^{-n}b_{cv}(1) \\
0 & -(A+\lambda I)^{-1}b_{cv}(2) & \ddots & \vdots \\
\vdots & \vdots & \ddots & -(A+\lambda I)^{-2}b_{cv}(n-1) \\
0 & 0 & \cdots & -(A+\lambda I)^{-1}b_{cv}(n) \\
v_1 & v_2 & \cdots & v_n\end{bmatrix}$$
Gramian Matrix

Where \( I \) is \( n \times n \) identity matrix and \( 0 \) is a \( n \times 1 \) zero vector.

**Example:**

\[
\dot{x} = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} x + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \\
y = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix} x
\]

**Lemma 1:** Let \( W_{ij} \) be the cross-gromians matrix for the elementary subsystem \((A, b_i, c_i, 0)\) then the original multivariable system cross-gramian matrix is

\[
W_{co} = \sum_{i} W_{ci}^{ii}
\]

Let \((A, B, C)\) be any minimal realization of the linear stable \( m \times m \) multivariable plant \( G(s) \). Also \((A, b_i, c_i, 0)\) are the elementary subsystems defined for \( i, j = 1, 2, ..., m \). Each subsystems has a corresponding cross-gramian matrix and the norm of this matrix, the largest singular value \( \sigma(W_{co}^{ij}) \), is employed to quantify the ability of input \( u_i \) to control \( y_j \). The following matrix is now defined:

\[
\Gamma = [\Gamma_{ij}] = [\sigma(W_{co}^{ij})]
\]
If $G_{ij} = 0$ for a given pair $(i,j)$, then $W_{oo}^{ij} = 0$, leading to $\Gamma_{ij}$. This implies that a block diagonal $G$ gives a block diagonal $\Gamma$ matrix, with the same structure.

It is important to observe that $\Gamma$ takes the full dynamic effects of the system into account and not only the steady-state performance or the behavior at a single frequency, as in RGA. In other words, $\Gamma$ matrix can be used as an interaction measure for the linear multivariable plant. Note that, to compute each element of $\Gamma$ matrix, only one matrix equation should be solved. This considerably reduces the computational task of the methodology in comparison with the two Lyapunov equations to compute the controllability and observability gramian matrices as in other methods.

**Algorithm:** Input-output pair selection for stable linear multivariable systems:

1. **1**st step: calculate the cross-Gramian matrix for each SISO elementary subsystem.
2. **2**nd step: compute the largest singular value of each cross-gramian matrices and compute the Dynamical input-output pairing matrix.
3. **3**rd step: find the largest value in each row of matrix $\Gamma$, which corresponds to the appropriate input-output pair.
Gramian Matrix

Any minimal and stable space realization of the plant can be used in the proposed algorithm. This makes the algorithm invariant under state space realizations.

\[
[\Gamma_{ij}] = [\bar{\sigma}(w_{co}^{ij})] = [\sqrt{\lambda_{\text{max}}((w_{co}^{ij})^2)}]
\]

Also, it is straightforward to show that the largest singular value of the cross-gramian matrix of the balanced realization is equivalent to the maximum of the absolute values of the eigenvalues of the cross-gramian for any realization as:

\[
\bar{\sigma}(w_{co}) = \sqrt{\lambda_{\text{max}}((w_{co})^2)} = \max \{ |\lambda(w_{co})| \}
\]

Example 1:

\[
G(s) = \begin{bmatrix}
\frac{-0.9019s + 15.47}{s^2 + 9.163s + 15.47} & \frac{-3.327}{s + 6.931} \\
\frac{0.8926}{s + 2.231} & \frac{0.7549s + 13.92}{s^2 + 9.163s + 15.47}
\end{bmatrix}
\]

\[
\text{RGA}(G) = \begin{bmatrix}
0.8242 & 0.1758 \\
0.1758 & 0.8242
\end{bmatrix}
\]

\[
\Gamma = \begin{bmatrix}
1.0070 & 0.2000 \\
0.2400 & 0.8647
\end{bmatrix}
\]

Have same result \((u_1 - y_1, u_2 - y_2)\).
Gramian Matrix

Example 1:

\[
G(s) = \begin{bmatrix}
\frac{1.073}{s+1} & \frac{4.2s+1.994}{s^2+3s+1.948} \\
-2s-1.003 & 1 \\
\frac{s^2+1.3s+0.9712}{s^2+1.3s+0.9712} & \frac{1}{s+1}
\end{bmatrix}
\]

\[
RGA(G) = \begin{bmatrix}
0.5034 & 0.4966 \\
0.4966 & 0.5034
\end{bmatrix}
\]

\[
\Gamma = \begin{bmatrix}
0.5362 & 0.7787 \\
0.9020 & 0.4998
\end{bmatrix}
\]

Have different result, but which of them is better?

Wittenmark and Salgado showed that use of RGA method to paring can give an unstable closed loop system. this new method has superior performance when the multivariable interaction has a non-monotonic behavior in frequency.

It is also simpler than other indices like Hankel interaction index and gramian-based interaction measure, since it does not require computation of both controllability and observability grammian matrices. and only computes a cross-gramian matrix for each elementary subsystem.
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RGA Analysis of Uncertain Multivariable Plants

There are different approaches to input-output selection and RGA is the first and the most widely used analytical tool for this problem. However, the proposed approaches are mainly applicable to known multivariable plants and fail in the presence of plant uncertainties. The attempts to overcome the uncertainty problem in process models can only partially solve the issue and cannot identify the changes in the input-output selection.

We show that, The parametric model uncertainty is considered and a graphically based test is presented to identify the possible input-output selection changes resulting from the parameter parameter changes.

- Consider a linear multivariable plant described by a transfer function matrix $G(s)$ with $m$ inputs and outputs. The RGA matrix has been defined as follow.

$$\Gamma = G(0) \otimes G(0)^{-T}$$  \hspace{1cm} (1)
and each element of $\Gamma$ ($\lambda_{ij}$) can be calculated by

$$\lambda_{ij} = \frac{(-1)^{i+j} g_{ij} \det(G^{ij})}{\det(G)} \quad i, j = 1, 2, \ldots, m$$  \hspace{1cm} (2)

**2.1 Two input - Two output multivariable plants**

In the case of multivariable plants with two inputs and two outputs, equation (2) can be rewritten as

$$\lambda_{11} = \frac{g_{11}g_{22}}{g_{11}g_{22} - g_{12}g_{21}}$$  \hspace{1cm} (3)

and by defining the following variable

$$k = \frac{g_{12}g_{21}}{g_{11}g_{22}}$$  \hspace{1cm} (4)

equation (3) can be written as

$$\lambda_{11} = \frac{1}{1 - k}$$  \hspace{1cm} (5)

and, similarly $12 \lambda$ can be written as follows

$$\lambda_{12} = \frac{-k}{1 - k}$$  \hspace{1cm} (6)
RGA Analysis of Uncertain Multivariable Plants

Since, in such multivariable plants the input-output pairing can be determined from the comparison of the elements of the first row of the RGA, \( \lambda_{11} \) and \( \lambda_{12} \) are compared. Let \( \lambda_{11} > \lambda_{12} \), in this case it follows from equations (5) and (6) that

\[
-1 < k < 1
\]  

(7)

In the face of parametric uncertainties, \( k \) will be an uncertain parameter and can be present if \( \Delta k \) causes a change in the previous input-output pairing, i.e. \( \lambda_{12} > \lambda_{11} \), then

\[
\frac{1}{1-k'} < \frac{-k'}{1-k'}
\]  

(9)

\[
k' > 1 \text{ or } k' < -1
\]  

(10)

Equations (7) and (10) are now presented on an axis, as is shown in fig1. A transfer from region 1 to 2 or vice versa shows a change in the input-output pairing due to the parametric uncertainties.
RGA Analysis of Uncertain Multivariable Plants

• Example 1:

\[ G(s) = \begin{bmatrix} \frac{(12.8 + \delta_{11})e^{-s}}{10.9s + 1} & -\frac{(18.9 + \delta_{12})e^{-3s}}{14.4s + 1} \\ \frac{(6.6 + \delta_{21})e^{-7s}}{21s + 1} & -\frac{(19.4 + \delta_{22})e^{-3s}}{10.9s + 1} \end{bmatrix} \]

Where \( \delta_{11} \in [-6.2], \ \delta_{12} \in [-7.3], \ \delta_{21} \in [-1.3], \ \delta_{22} \in [-2.2], \) and its corresponding RGA for the nominal case \( \delta_{ij} = 0 \ (i, j = 1, 2) \) is and \( k = 0.5023. \)

\[ \Gamma_1 = \begin{bmatrix} 2.01 & -1.01 \\ -1.01 & 2.01 \end{bmatrix} \]

The RGA matrix shows \((u_1 - y_1, u_2 - y_2)\) is an appropriate input-output pairing. For \( \delta_{11} = -4, \ \delta_{12} = -5, \ \delta_{21} = 1, \ \delta_{22} = 0, \) \( k \) is changed to

\[ k' = \frac{(6.6 + 1)(-18.9 - 5)}{(12.8 - 4)(-19.4)} = 1.064 > 1 \]

Which clearly indicates as is shown in Fig 1 that a change in input-output pairing has occurred. This result is also verified by calculating the RGA matrix for the new parameters, i.e.

\[ \Gamma_2 = \begin{bmatrix} -15.63 & 16.63 \\ 16.63 & -15.63 \end{bmatrix} \]
2.2 Three input - Three output multivariable plants

In the case of multivariable plants with three input and output, equations (2) can be rewritten as

\[
\lambda_{11} = \frac{g_{11} \det(G^{11})}{g_{11} \det(G^{11}) - g_{12} \det(G^{12}) + g_{13} \det(G^{13})}
\]

\[
\lambda_{12} = \frac{-g_{12} \det(G^{12})}{g_{11} \det(G^{11}) - g_{12} \det(G^{12}) + g_{13} \det(G^{13})}
\]

\[
\lambda_{13} = \frac{g_{13} \det(G^{13})}{g_{11} \det(G^{11}) - g_{12} \det(G^{12}) + g_{13} \det(G^{13})}
\]

and by defining the following variables

\[
k_1 = \frac{g_{12} \det(G^{12})}{g_{11} \det(G^{11})}, \quad k_2 = \frac{g_{13} \det(G^{13})}{g_{11} \det(G^{11})}
\]

the element of the RGA matrix can be written as

\[
\lambda_{11} = \frac{1}{1 + k_2 - k_1}
\]

\[
\lambda_{12} = \frac{-k_1}{1 + k_2 - k_1}
\]

\[
\lambda_{13} = \frac{k_2}{1 + k_2 - k_1}
\]

The regions in the \((k_1, k_2)\) coordinates which indicate a change in the input-output pairings are now determined.
Case I: \((\lambda_{11} > \lambda_{12}, \lambda_{11} > \lambda_{13})\)

In this case, equations (13) give a closed region characterized by

\[
1 + k_2 - k_1 > 0 \\
k_1 > -1 \\
k_2 < 1
\]

and is shown in fig 2. The shaded region in fig 2 represents the inequalities in (13) and shows that \((u_1 - y_1, u_2 - y_2)\) are an appropriative pairing, but a final decision must be made after considering the other cases.
Case II: \( (\lambda_{12} > \lambda_{11}, \lambda_{12} > \lambda_{13}) \)
equations (13) give a closed region characterized by (15) And is shown in fig 3:

\[
\begin{align*}
1 + k_2 - k_1 & > 0 & 1 + k_2 - k_1 & < 0 \\
 k_1 & < -1 & k_1 & > -1 \\
 k_2 + k_1 & < 0 & k_2 + k_1 & > 0
\end{align*}
\] (15)

Case III: \( (\lambda_{13} > \lambda_{11}, \lambda_{13} > \lambda_{12}) \)
Similarly, equations (13) give

\[
\begin{align*}
1 + k_2 - k_1 & > 0 & 1 + k_2 - k_1 & < 0 \\
 k_2 & > 1 & k_2 & < 1 \\
 k_2 + k_1 & > 0 & k_2 + k_1 & < 0
\end{align*}
\] (16)

and this region is shown in fig 4
Test procedure

1) Determine the $\hat{k}_i = (k_1, k_2)$ variables using the following equation.

$$\hat{k}_i = \frac{\Gamma(\cdot, 2).\Gamma(\cdot, 3)}{\Gamma(\cdot, 1).\Gamma(\cdot, 1)}$$ (17)

2) Identify the points $\hat{k}_1$, $\hat{k}_2$, and $\hat{k}_3$ in the $k_2 - k_1$ diagram.

3) If each of the $\hat{k}_1$, $\hat{k}_2$, and $\hat{k}_3$ lie in one of the three regions, then we can use decentralize control.

4) A shift of the indices $\hat{k}_1$, $\hat{k}_2$, and $\hat{k}_3$ from one region to another, indicates a change in the input-outputs pairing.

Example 2:

$$G(s) = \begin{bmatrix}
   0.66e^{-2.6s} & -0.61e^{-3.5s} & -0.0049e^{-s} \\
   6.7s + 1 & 8.64s + 1 & 9.06s + 1 \\
   1.11e^{-0.65s} & -2.36e^{-3s} & -0.012e^{-1.2s} \\
   3.25s + 1 & 5s + 1 & 7.09s + 1 \\
   -33.68e^{-9.2s} & 46.2e^{-9.4s} & 0.87(11.61s + 1)e^{-2s} \\
   8.15s + 1 & 10.9s + 1 & (3.89s + 1)(18.8s + 1)
\end{bmatrix}$$
RGA Analysis of Uncertain Multivariable Plants

Case I:

\[
G_1 = G(0) + 0.00 \times \begin{bmatrix}
-1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1
\end{bmatrix} \otimes G(0)
\]

Fig 5 represents the nominal case and its corresponding RGA is:

\[
\hat{r}_1 = \begin{bmatrix}
1.945 & -0.673 & -0.272 \\
-0.664 & 1.899 & -0.235 \\
-0.281 & -0.225 & 1.506
\end{bmatrix}
\]

Case II:

\[
G_2 = G(0) + 0.05 \times \begin{bmatrix}
-1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1
\end{bmatrix} \otimes G(0)
\]

Fig 6 shows the position of \( \hat{k}_1, \hat{k}_2 \) and \( \hat{k}_3 \) indicates that a change in the input-output pairing has not occurred. This is verified by the corresponding RGA matrix given by

\[
\Gamma_2 = \begin{bmatrix}
1.9913 & -0.7306 & -0.2607 \\
-0.6371 & 1.8213 & -0.1842 \\
-0.3542 & -0.0907 & 1.4449
\end{bmatrix}
\]

Case III:

\[
G_3 = G(0) + 0.3 \times \begin{bmatrix}
-1 & 1 & -1 \\
1 & -1 & -1 \\
1 & -1 & 1
\end{bmatrix} \otimes G(0)
\]

Fig 7 shows the position of \( \hat{k}_1, \hat{k}_2 \) and \( \hat{k}_3 \) indicates that a change has occurred in the input-output pairing. This is also verified by the corresponding RGA matrix given by

\[
\Gamma_3 = \begin{bmatrix}
-2.0905 & 2.8411 & -0.2495 \\
3.2140 & -1.7431 & -0.4708 \\
-0.1234 & -0.0979 & 1.2214
\end{bmatrix}
\]
RGA Analysis of Uncertain Multivariable Plants

Fig 5

Example 2: Case I

Fig 6

Example 2: Case II

Fig 7

Example 2: Case 3
RGA Analysis of Uncertain Multivariable Plants

The present input-output pairing methods can easily fail in the case of parametric uncertainties. In this paper, a test procedure has been proposed to nominate the appropriate input-output pairings in the face of parametric plant uncertainties. The regions indicating these pairings are shown graphically to further assist the designer in an input-output selection process. Examples have been provided to show the effectiveness of the proposed.
Topics to be covered include:

- Decentralized control
- RGA
  - DRGA
  - ROmA
  - REGA
  - RNGA
- Gramian Matrix
- RGA Analysis of Uncertain Multivariable Plants
- ARGA
ARGA Loop Pairing Criteria for Multivariable SYS

The new index, called ARGA (Absolute Relative Gain Array), solves the pairing problem taking into account the bounds of the absolute stability in the presence of dynamical interactions.

ARGA INDEX: DEFINITION

The first step for defining ARGA index is the characterization of the nonlinear components. A failure condition in a loop is modeled by the diagonal matrix $N$ of nonlinear terms as in the hypotheses of the circle theorems. Therefore the nonlinear components are described via an algebraic input/output function within a limited sector. Note that this approach includes all the cases due to loop changes originated by saturations in the loop or by manual exclusions of single loops for maintenance purposes. A loop shut down can be viewed as the transition from a condition of a short circuit (output equals input) to a condition of an open circuit (output equals zero) in an electrical circuit. Therefore if a loop has a ‘soft’ or ‘hard’ disconnection, it can be represented by a non linear function confined within a sector $[0, k]$. 
Consider now a MIMO system, reachable, observable and open loop stable, described by the $n \times n$ transfer function matrix $G(s) = \{g_{ij}(s)\}$.

For each element $g_{ij}(s)$ of the transfer function matrix the critical frequency $\omega_{\pi,ij}$ and the limit gain for absolute stability $k_{ij}$ are evaluated in the hypothesis of non interacting loop and of nonlinearities confined within a sector $[0, k]$.

This choice stems from the observation that in single –input single-output systems the critical frequency $\omega_{\pi,ij}$ (rad/s) remains unchanged in the passage from open loop to closed loop. This property holds also for MIMO systems, whenever perfect decoupling occurs. Then we use critical frequencies in the passage from open loop to closed loop, for measuring interactions in MIMO systems.

Difficulties in critical frequencies computation may eventually arise, since $\omega_{\pi,ij}$ does not necessarily exist or is computable. In such cases, an additional time delay may be inserted in all channels.

A preliminary evaluation of the critical frequency $\omega_{\pi,ij}$ and of the limit gain $k_{ij}$ in case of circle criterion may be estimated from the Nyquist plot of the frequency response $g_{ij}(s)$, as shown in following fig.
A second step of the method estimates the interactions of the other loops on the i-j channel. In the frequency domain the method of Cook [9] is considered. It employs the symmetric Gershgorin bands, for quantifying interactions. Gershgorin bands are superimposed on Cook circles centered on the appropriate point of each diagonal locus of the linear system \( G(j\omega) \).

Absolute stability is guaranteed if the Gershgorin bands do not contain or intersect critical circles in the Rosenbrock sense, i.e., in case of open loop stable systems each critical circle must be external to the Gershgorin band.

Cook circles are built to verify a dominance condition defined as:

\[
R_{ij} (\bar{\omega}) = \frac{1}{2} \left\{ \sum_{k=1}^{n} \left| g_{ik} (j\bar{\omega}) \right| + \sum_{k=1}^{n} \left| g_{kj} (j\bar{\omega}) \right| \right\}
\]

A new performance index is therefore introduced as:

\[
a_{ij} = \omega_{\pi,ij} \cdot k_{ij}
\]

A new evaluation of the critical frequency \( \omega_{\pi,ij} \) and of the limit gain \( k_{ij} \) is then performed.
A graphical interpretation of the Cook method for extracting the limit gain is shown in Fig. 2. The critical frequency $\omega_{\pi,ij}$ is evaluated as the frequency for which the corresponding Cook circle intersects the negative real axis most on the left (see fig. 3).

This result is quite conservative, and it represents the worst case, when the interaction of the other loops on stability is maximum: this consideration leads to the safest choice of pairings.

A product based on the interaction measurement is:

$$f_{ij} = \bar{\omega}_{\pi,ij} \cdot \bar{k}_{ij}$$
ARGA

By mimicking the RGA procedure, the products (3) are considered for creating a new matrix $F = \{f_{ij}\}$ and the pairings can be easily verified introducing the matrix ARGA, in a way analogous to the RGA definition:

$$\Psi = F \otimes F^{-T}$$

The most important rules for pairings can be summarized as:
1. elements of ARGA matrix closest to 1 suggest the preferred pairings
2. all elements of ARGA matrix chosen for pairings must be positive
3. elements of ARGA matrix with values much greater than 1 should be considered indices of incorrect pairing

**Example 1:**

$$\Lambda = \begin{bmatrix} 2.0094 & -1.0094 \\ -1.0094 & 2.0094 \end{bmatrix}$$

$$G(s) = \begin{bmatrix} 12.8e^{-s} & -18.9e^{-3s} \\ 16.7s+1 & 21s+1 \\ 6.6e^{-7s} & -19.4e^{-3s} \\ 10.9s+1 & 14.4s+1 \end{bmatrix}$$

$$\Psi = F \otimes F^{-T} = \begin{bmatrix} 1.005 & -0.005 \\ -0.005 & 1.005 \end{bmatrix}$$

Pairings suggested by the RGA rule and both of them suggest the use of a diagonal pairing ($y1-u1$, $y2-u2$) in good agreement with the physical behaviour of the process. ARGA index confirms such pairing also in terms of integrity and of absolute stability.
ARGA

Example 2:

\[
G(s) = \begin{bmatrix}
\frac{e^{-s}}{1+s} & 1 \\
-1 & e^{-2s} \\
\frac{1}{1+s} & \frac{1}{1+s}
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
0.5 & 0.5 \\
0.5 & 0.5
\end{bmatrix}
\]

\[
\Psi = F \otimes F^{-T} = \begin{bmatrix}
-0.1417 & 1.1417 \\
1.1417 & -0.1417
\end{bmatrix}
\]

ARGA index confirms the off-diagonal pairing suggested in terms of integrity and of absolute stability, where the steady-state RGA does not suggest any preferential pairing.

Example 2:

\[
G(s) = \begin{bmatrix}
\frac{2.5e^{-5s}}{(1+15s)(1+2s)} & 1 \\
\frac{1}{1+4s} & -4e^{-5s} \\
\frac{1}{1+3s} & \frac{1}{1+20s}
\end{bmatrix}
\]

\[
\Lambda = \begin{bmatrix}
0.9091 & 0.0909 \\
0.0909 & 0.9091
\end{bmatrix}
\]

\[
\Psi = F \otimes F^{-T} = \begin{bmatrix}
-0.0011 & 1.0011 \\
1.0011 & -0.0011
\end{bmatrix}
\]

RGA suggests the use of a diagonal pairing \((y_1-u_1, y_2-u_2)\), but the ARGA leads to off-diagonal pairing \((y_1-u_2, y_2-u_1)\). But which is better?
In the following figures Nyquist diagrams including Cook circles which define the Gershgorin bands are represented, for helping the reader to visualize the dominance properties.
From Fig. 4 and Fig. 5 it can be shown how large are the margins of absolute stability in both loops chosen for pairing in case of example 3. In bold line are represented the limit cases for the evaluation of the critical parameters $\omega \pi , ij$ and $kij$. The sectors of absolute stability are: $[0, 2.5141]$ for the element 1-2 and $[0, 2.974]$ for the element 2-1. In Fig. 6 and Fig. 7 the case of incorrect diagonal pairing is represented. The sectors of absolute stability are: $[0, 0.635]$ for the element 1-1 and $[0, 0.617]$ for the element 2-2. Comparing the sectors in case of off-diagonal and diagonal pairings, it can be noted that the off-diagonal pairing is the correct choice in terms of absolute stability integrity and therefore of integrity to abnormal operating conditions.
PROJECTS:

**Project #1**: Consider the process described by the following matrix transfer function:

\[
G(s) = \frac{1}{(s+1)(2s+1)^2(0.5s+1)} \begin{bmatrix}
0.5 & -0.6 & 0.1 \\
0.2 & 0.8 & 0.3 \\
-1.0 & 0.1 & 1.0
\end{bmatrix}
\]

**A)** Evaluate steady-state RGA.

**B)** By using ARGA method, we know \( F \), so compute \( \psi \).

**C)** What is the appropriate pairing in each case?

**D)** By using Matlab software, show both results of input output pairing.

**Hint #1**: For part (D), you should use decentralized controllers, diagonal controller for diagonal pairing & off-diagonal controller for off-diagonal pairing.

**Hint #2**: Controller \( K(s) \) can be in the following structure for diagonal pairing, \( a \) & \( b \) can be PI or PID controllers.

\[
K(s) = \begin{bmatrix}
\alpha & 0 \\
0 & \beta
\end{bmatrix}
\]
PROJECTS:

- **Project #2**: Consider the following plant.
  - A) Compute RGA.
  - B) For following F compute $\psi$.
  - C) Which kind of pairing is appropriate?
  - D) By using Matlab software, compare diagonal & off diagonal pairing.
  - E) What these results imply? Change coefficients until satisfying results arise.

- **Hint #1**: Use following controllers in a diagonal / off-diagonal matrix structure.
  - $K_1 = \begin{bmatrix} 0.5(1+1/(100s)) & 0 \ 0 & 0.5(1+1/(100s)) \end{bmatrix}$
  - $K_2 = \begin{bmatrix} 0 & 0.25(1+1/(1)) \ 0.5(1+1/(10)) & 0 \end{bmatrix}$

- $G(s) = \begin{bmatrix} \frac{e^{-s}}{1+s} & \frac{1}{1+s} \\
-1 & \frac{e^{-2s}}{1+s} \end{bmatrix}$

- $F = \begin{bmatrix} 0.7778 & 1.7663 \\
1.7663 & 0.4977 \end{bmatrix}$
PROJECTS:(Arbitrary)

- **Project#3:** Consider the following plant.
  - A) compute RGA.
  - B) compute $\Phi$.
  - C) Which kind of pairing is appropriate?
  - D) By using Matlab software, compare diagonal & off diagonal pairing.
  - E) What these results imply? change coefficients until satisfying results arise.

**Hint#1:** use following controllers in a diagonal / off-diagonal matrix structure.
- $K_1 = [0.5(1+1/(100s)) \ 0 \ 0.5(1+1/(100s))]$;
- $K_2 = [0 \ .25(1+1/(10s));0.5(0.1+1/(10s)) \ 0]$;

**Hint#2:** off-diagonal pairing leads to unstable results, try to reach stable results by changing coefficients, or using another controller such as PID controller instead of PI.
References

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Thanks for your attention