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AN EXTENSION OF A TERNARY DERIVATION ON A HILBERT C^*-MODULE

GHOLAMREZA ABBASPOUR TABADKAN

Abstract. Let E be a Hilbert module over a C^*-algebra B. A *ternary derivation* of E is a densely defined linear map $\delta : D(\delta) \subset E \to E$ that fulfills :

$$\delta(x\langle y,z\rangle) \;=\; \delta(x)\langle y,z\rangle + x\langle\delta(y),z\rangle + x\langle y,\delta(z)\rangle \qquad (x,y,z\in E),$$

where $D(\delta)\ \langle\, D(\delta)\, ,\, D(\delta)\,\rangle \subset D(\delta)$, that is, $D(\delta)$ is invariant under the ternary product $(x,y,z)\mapsto x\langle y,z\rangle$. In this talk we extend each ternary derivation δ of a full Hilbert B-module E to a derivation Δ on the linking algebra of E and investigate the relation between δ and Δ. In particular we show that δ is a generator of a dynamical system on E if and only if Δ is a generator of a C^*-dynamical system on the linking algebra.

1. Introduction.

A *Hilbert B-module* is a right module E over a C^*-algebra B with a sesquilinear inner product $\langle .,.\rangle : E\times E \to B$ that satisfies:
(i) $\langle x, yb\rangle = \langle x,y\rangle b \quad (x,y\in E, b\in B)$.
(ii) $\langle x,x\rangle \geq 0$ *and* $\langle x,x\rangle = 0$ *iff* $x=0$ $(x\in E)$.
(iii) E is complete with respect to the norm $\|x\| := \|\langle x,x\rangle\|^{\frac{1}{2}}$.
A Hilbert B-module E is *full*, if the *range ideal* $B_E := span\langle E,E\rangle$ is dense in B.

By $B^a(E)$ and $K(E)$ we denote the C^*-algebra of all *adjointable* operators and of all *compact* operators on E, where $K(E)$ is the completion of the $*$-algebra $F(E)$ of *finite-rank* operators which is spanned by the *rank-one* operators xy^* from E to E defined by $z\mapsto x\langle y,z\rangle \ \ (x,y,z\in E)$.

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Let E be a Hilbert B-module. A *generalized unitary* on E is a surjection linear map u on E that fulfills

$$\langle ux, uy \rangle = \varphi(\langle x, y \rangle) \ \ (x, y \in E)$$

for some automorphism φ of B. We also say u is a φ-unitary. A *generalized derivation* of E is a densely defined linear map $\delta : D(\delta) \subset E \to E$ that fulfills

$$\delta(xb) = \delta(x)b + xd(b) \ \ x \in D(\delta), \ \ b \in D(d)$$

for some derivation d of B, which $D(\delta)$ is a right $D(d)$-module. In [1] a *dynamical system* on a Hilbert B-module E describe as a strongly continuous one-parameter group $u = (u_t)_{t\in\mathbb{R}}$ of generalized unitaries and authors showed that the generator of a dynamical system is a generalized derivation. A *ternary derivation* of E is a densely defined linear map $\delta : D(\delta) \subset E \to E$ that fulfills :

$$\delta(x\langle y, z\rangle) \ = \ \delta(x)\langle y, z\rangle + x\langle \delta(y), z\rangle + x\langle y, \delta(z)\rangle \ \ x, y, z \in E$$

Where $D(\delta) \ \langle \, D(\delta) \, , \, D(\delta) \, \rangle \ \subset \ D(\delta)$, that is, $D(\delta)$ is invariant under the ternary product $(x, y, z) \mapsto x\langle y, z\rangle$.

2. Main results

Theorem 2.1. *Every ternary derivation δ of a full Hilbert B-module E is a generalized derivation. More precisely, there is a unique derivation d_δ of B with the dense domain $D(d_\delta) := span\langle D(\delta), D(\delta)\rangle$ that fulfills*

$$d_\delta(\langle x, y\rangle) = \langle \delta(x), y\rangle + \langle x, \delta(y)\rangle$$

d_δ turns δ into a d_δ-derivation. Moreover, d_δ is a $*$-derivation.

Corollary 2.2. *Every generator of a dynamical system on a full Hilbert module is a ternary derivation.*

Now suppose that δ is a ternary derivation of the full Hilbert B-module E. We want to extend δ to a derivation of *linking algebra* $\begin{pmatrix} B & E^* \\ E & K(E) \end{pmatrix}$.

By Theorem [2.1] δ determines the $*$-derivation d_δ of B which is the candidate for how to extend δ to the corner B of the linking algebra. To find the extension to $K(E)$, we observe that E^* is a full Hilbert $K(E)$-module, the *dual module* of E, with inner product $\langle x^*, y^*\rangle := xy^* \in K(E)$ and the right action $x^*a := (a^*x)^*$ of elements $a \in K(E)$. Of course, $\delta^*(x^*) := \delta(x)^*$ defines a ternary derivation of E^* with domain $D(\delta^*) := D(\delta)^*$, and by Theorem [2.1] there is a unique $*$-derivation d_{δ^*} of $K(E)$ defined on the domain $D(d_{\delta^*}) = D(\delta)D(\delta)^*$, turning δ^* into a d_{δ^*}-derivation. It is routine to check that

$$\begin{pmatrix} b & y^* \\ x & a \end{pmatrix} \mapsto \begin{pmatrix} d_\delta(b) & \delta^*(y^*) \\ \delta(x) & d_{\delta^*}(a) \end{pmatrix}$$

defines a $*$-derivation Δ of the linking algebra. This leads us to the following corollary:

Corollary 2.3. *Every ternary derivation of a full Hilbert module extends as a $*$-derivation to the linking algebra.*

Now we can investigate the relation between δ and the extension Δ, for example we will show that ternary derivation δ is a pre-generator of a dynamical system of E if and only if Δ is a pre-generator of a C^*-dynamical system of linking algebra of E.

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HARMONIC ANALYSIS NOW

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ABSTRACT. This short note gives an overview of the Harmonic Analysis on groups and group-like structures. It is quite a personal view and covers only a selection of topics, but I have tried to present a good selection of recent advances on each topic.

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A CLASS OF NON-NORMABLE STAR TOPOLOGICAL ALGEBRAS

ESMAEIL ANSARI-PIRI

Abstract. The well known simple relation between the norm of an element x and xx^* in a C^*-algebra has a high potential and is too much important. In this note we try to extend this relation to metrizable -not necessarily locally convex - star algebras.

Using the new equivalent definition in the absence of norms and semi-norms we get some simple results.

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COHERENT STATES OF GENERAL SEMIDIRECT PRODUCT GROUPS

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Abstract. We develop the continuous theory of coherent states associated to group representation. Most of the interesting groups encountered in the physics literature are semidirect product groups. Recently Ali, Antoine and Gazeau [1] have produced coherent states on the some of these groups. We present some of their calculations in a more general setting.

1. Introduction

Given a locally compact group or a Lie group G, choose a square integrable representation U of G (if one such exists) in some Hilbert space $\mathcal{H}$, and fix a vector $\eta \in \mathcal{H}$ (possible satisfying an additional admissibility condition). Then a family of coherent states, associated to U, is defined to be the set of vectors

$$\{\eta_g = U(g)\eta \ ; \ g \in G\},$$

in $\mathcal{H}$. In other words, coherent states are the elements of the orbit of η under the (square integrable) representation U.

In [1] Ali, Antoine and Gazeau discuss a general analysis of coherent states on the semidirect product group of the type $S \times V$, where V is an n-dimensional real vector space and S is usually a subgroup of $GL(V)$ (the group of all nonsingular linear transformation of V). The present paper proposes another method to extend the concept of coherent state. This method can be applied to a general semidirect product group $H \times_\tau K$, where H and K are locally compact groups and K is also abelian.

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2. Quasi Invariant Measure on a Homogeneous Space

Let X is a $G-$space. A Borel section on X is a map $\sigma : X \longrightarrow G$, satisfying $\pi(\sigma(x)) = x$, for all $x \in X$, where $\pi : G \longrightarrow X$ is the canonical mapping. This is always the case if G is second countable. Let ν be a quasi-invariant measure on X and σ a Borel section. To generate coherent states we require another quasi-invariant measure ν_σ, which in a sense is the standard quasi-invariant measure for the chosen section. Indeed we define

$$d\nu_\sigma(x) = \lambda(\sigma(x), x)d\nu(x). \tag{2.1}$$

For the remainder of this section assume that $h \longmapsto \tau_h$ be a homomorphism of H into Aut(K) and $H \times_\tau K$ is the semidirect product of H and K, respectively. The left Haar measure of G is $d\mu_G(h,k) = \delta(h)d\mu_H(h)d\mu_K(k)$, in which δ is a positive continuous homomorphism on H and is given by

$$\mu_K(E) = \delta(h)\mu_K(\tau_h(E)), \tag{2.2}$$

for all measurable subsets E of K,(15.29 of [4]).

Theorem 2.1. *Let $G = H \times_\tau K$ be semidirect product of H and K. Take $X = G/\widetilde{H}$, where $\widetilde{H} = \{(h, 1_K) \ ; \ h \in H\}$ is the closed subgroup of G and fix a Borel section σ. Then there is a unique quasi invariant measure ν_σ such that $d\nu_\sigma(x) = \delta(\sigma_1(x)^{-1})d\nu(x)$. Moreover ν_σ is an invariant measure iff $\delta \equiv 1$.*

3. Covariant Coherent States

Let G be a locally compact group and U a continuous unitary irreducible representation of G on a separable Hilbert space $\mathcal{H}$. Now we define coherent states of U associated to a Borel section σ of G and introduce the notion of square integrability $mod(H,\sigma)$ for U, where H is a closed subgroup of G. Then the continuous wavelet transform on G can be extended to a general form.

Definition 3.1. Suppose that X is a homogeneous space of G, $X = G/H$, equipped with a strongly quasi invariant measure ν. Also let $\sigma : X \longrightarrow G$ be a Borel section. Then $\eta \in \mathcal{H}$ is called *admissible* $mod(H,\sigma)$ when

$$\int_X | < U(\sigma(x))\eta, \phi > |^2 d\nu_\sigma(x) < \infty, \qquad \forall \phi \in \mathcal{H}.$$

In this case, the set $\mathcal{H}_\sigma = \{U(\sigma(x))\eta; \ x \in X\}$ is said to be as a family of *covariant coherent states* for U. Let $F = |\eta >< \eta|$, $F_\sigma(x) = U(\sigma(x))FU(\sigma(x))^\star$, and suppose that

$$\int_X F_\sigma(x)d\nu_\sigma(x) = A_\sigma \qquad A_\sigma, A_\sigma^{-1} \in \mathcal{L}(\mathcal{H}),$$

the integral converging weakly. Then the coherent states $\mathcal{H}_\sigma$ are called square integrable and the representation U is said to be *square integrable* $mod(H,\sigma)$. It is easy to see that U is square integrable $mod(H,\sigma)$ iff η is admissible $mod(H,\sigma)$. If $U(h)FU(h)^\star = F$ then A_σ is a multiple of identity iff X admits a left invariant measure.

Definition 3.2. The quasi regular representation $(U, L^2(K))$ associated to the semidirect product group $G = H \times_\tau K$ is defined by

$$U(h,k)f(y) = \delta(h)^{\frac{1}{2}} f(\tau_{h^{-1}}(yk^{-1})) \ , \ f \in L^2(K) \text{ and } (h,k) \in G, \ y \in K.$$

In [2] it is shown that if there exists $\psi \in L^2(K)$ such that

$$C_\psi^2 := \int_H |\widehat{\psi}(\omega \circ \tau_h)|^2 d\mu_H(h) < \infty, \tag{3.1}$$

then U is square integrable.

Theorem 3.3. *Let $(U, L^2(K))$ be the quasi regular representation on $G = H \times_\tau K$. Take $X = G/\widetilde{H}$ and fix a Borel section σ, also let the quasi invariant measure ν_σ introduced by (2.1). Then $\psi \in L^2(K)$ is admissible mod$(\widetilde{H}, \sigma)$ if*

$$\int_X \|\psi \circ \tau_{\sigma_1(x)^{-1}}\|_2 \ d\nu(x) < \infty$$

Suppose now that ψ is an admissible *mod*$(\widetilde{H}, \sigma)$ in $L^2(K)$, i.e. for all $\eta \in L^2(K)$ $\int_X | < U(\sigma(x))\psi, \eta > |^2 \ d\nu_\sigma(x) < \infty$. Then the generalized continuous wavelet transform

$$W_\psi : L^2(K) \longrightarrow L^2(X)$$

defined by

$$(W_\psi \eta)(x) = < U(\sigma(x))\psi, \eta >$$

is a unitary isomorphism.

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ON THE CLASS OF HEREDITARILY ℓ_1 BANACH SPACES WITHOUT THE SCHUR PROPERTY

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Abstract. We study X_α spaces constructed by Hagler and the first named author as examples of hereditarily ℓ_1 spaces failing the Schur property. For those spaces, relatively weakly compact subsets and complemented subspaces are discussed.

1. Definitions and Preliminaries

We use the standard terminology and usual notations in [4,5]. An infinite dimensional Banach space X is said to contains ℓ_1 hereditarily if every infinite dimensional subspace of X contains a subspace isomorphic to ℓ_1. We denote the natural projections associated with the unit vector basis by P_n. For a sequence (x_n) of elements of X we denote $[x_n]$ the closure of $span(x_n)$. If (x_n) is a Schauder basis for X, then bi-orthogonal functionals on X, associated to (x_n), is denoted by (x_n^*). A Banach space X is said to have the Schur property provided weak convergence of sequences in X implies their norm convergence.

The definition of X_α is given. First, by a block we mean an interval (finite or infinite) of integers. For any block F and $x = (t_1, t_2, ...)$ a finitely non-zero sequence of scalars, we let $< x, F >= \sum_{j\in F} t_j$. A sequence of blocks $F_1, F_2, \dots$ is admissible if $maxF_i < minF_{i+1}$ for each i. Finally, consider a sequence of nonnegative real (α_i) which satisfies the following properties:
(1) $\alpha_1 = 1$ and $\alpha_{i+1} \le \alpha_i$ for $i = 1, 2, ...,$
(2) $\lim_{i\to\infty} \alpha_i = 0$,
(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.
We now define a norm which uses the α_i's and admissible sequences of blocks in its

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definition. Let $x = (t_1, t_2, \ldots)$ be finitely non-zero sequence of reals, define

$$||x|| = \max \sum_{i=1}^{n} \alpha_i | < x, F_i > |$$

where the max is taken over all n, and admissible sequences $F_1, F_2, \ldots F_n$. The Banach space X_α is the completion of the finitely non-zero sequence of scalers in this norm.

Theorem 1.1. *Let X_α denote a specific space of the class, we have the following:*
(1) X_α is hereditarily ℓ_1.
(2) The sequence (e_i) is a normalized boundedly complete basis for X_α. Thus X_α is a dual space.
(3)(i) The sequence (e_i) is a weak cauchy sequence in X_α with no weak limit in X_α. In particular, X_α fails the Schur property. (ii) There is a subspace $\tilde{X}_\alpha$ of X_α which fails the Schur property , yet which is weakly sequentially complete.
(4) Let $B_1(X)$ denote the first Baire class of X_α in its second dual, i.e,

$$B_1(X) = \{x^{**} \in X^{**} : x^{**} \ \ is \ \ a \ \ weak^* \ \ limit \ \ of \ a \ sequence \ (x_n) \ in \ X_\alpha\}$$

Then $dim \frac{B_1(X)}{X} = 1$.

2. Main results

Now let $u_i = e_{2i} - e_{2i-1}$ and $\tilde{X}_\alpha = [u_i]$. Then

1) $\tilde{X}_\alpha$ is weakly sequentially complete,
2) (u_i) is an unconditional basis,
3) $u_i \longrightarrow 0$ weakly if $i \to \infty$.

The following theorem will give a necessary and sufficient condition for a subset of the Banach space $\tilde{X}_\alpha$ to be relatively weakly compact.

Theorem 2.1. *A subset $K \subseteq \tilde{X}_\alpha$ is relatively weakly compact iff,*

$$\forall \varepsilon > 0 \ \exists m \ \forall x \in K \ \exists F_x \subset N \ |F_x| = m :$$
$$||x - \sum_{j \in F_x} t_j u_j|| < \varepsilon$$

where $x = \sum_{j=1}^{\infty} t_j u_j$.

Definition 2.2. A subset Y of X is complemented in X if there is a bounded projection $P : X \longrightarrow Y$ such that $PX = Y$.

From the definition of the norm of X_α it can to see that the unit vector basis is spreading (equivalent to each of its subsequence) and bi-monotone. That is for each $x \in X_\alpha$ and $n < m$, $||(P_m - P_n)x|| \leq ||x||$. Observe each block F defines a functional which is bounded on X_α. In fact $< x, F > = \sum_{i \in F} e_i^*(x)$. Further, if (e_{i_k}) is a subsequence of (e_n), then $[(e_{i_k})]$ is complemented. Indeed if $\{F_i\}$ is a sequence of block without gaps ($MaxF_i + 1 = MinF_{i+1}$) such that $i_k \in F_k$, then $[\{e_{i_k}\}]$ is complemented by the projection

$$Px = \sum_{i=1}^{\infty} < x, F_k > .e_{i_k}$$

Lemma 2.3. *i) If $(x_i) \subset X_\alpha$ converges weak* to $x^{**} \in X^{**}$, then $x^{**} = x + \theta$ where $x \in X_\alpha$ and $e_i^*(\theta) = 0$ for all i.*
ii) If $(x_i) \subset X_\alpha$ is weakly cauchy, then (x_i) converges weak* to $x + \alpha\theta_0$ where $x \in X_\alpha$ and $\alpha = lim_{i\to\infty} < x_i - x, N >$, and θ_0 is the weak* limit of (e_i).

Lemma 2.4. *Let (x_i) be a block basic sequence of (e_i), let $F = \{M+1, M+2, ...\} \subset N$ and suppose $< x_i, F >= \gamma > 0$ for all i. Then for any scalars sequence (a_i),*

$$\gamma || \sum a_i e_i || \leq || \sum a_i x_i ||$$

Theorem 2.5. *If $T : X_\alpha \longrightarrow X_\alpha$ is bounded linear operator, then either TX_α or $(I - T)X_\alpha$ contains a complemented isomorphic of X_α.*

Theorem 2.6. *: Suppose $i : X \to Y, j : X^* \to Y^*$ are bounded linear operator such that $x^*(x) = (jx^*)(ix)$ for all $x \in X$ and $x^* \in X^*$.Then i, j are isomorphisms and if i is compact (weakly compact) then there exist a compact (weakly compact) projection corresponding to i on Y.*

Theorem 2.7. *Suppose Y_α and Y_β are the predual of X_α and X_β corresponding. Also $i : Y_\alpha \longrightarrow Y_\beta$, $j : X_\alpha \longrightarrow X_\beta$ are bounded linear maps such that $x(y) = (jx)(ix)$ for all $x \in X_\alpha, y \in Y_\beta$. If i is compact on Y_α, then there exist a compact linear operator on X_α corresponding to i.*

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VARIATIONAL PROBLEMS AND STRATIFIED REARRANGEMENTS

FARIBA BAHRAMI

Abstract. The Maximization and minimization of certain weakly continues convex or non-convex functionals, relative to the class of rearrangements of a given function have been considered by Burton. This paper concerns with stratified rearrangements of a function. It is shown here that the results of variationals principles on class of rearrangements are valid for class of stratified rearrangements.

1. Introduction

Study of variational problems on class of rearrangements is so important to prove the existence and stability of stationary vortices in two or three-dimensional flows. Benjamin considered this kind of variational problems in a theory of three dimensional vortex rings [1]. It has been used for existence proofs of family of three dimensional vortex-rings [5] and stable and stationary vortices in two dimensional flow at a localized seamount or mountain [2]. The main step to prove the existence of maximizer or minimizer for energy functional relative to rearrangements of a given functional on unbounded domain is to consider the problems on bounded domains. Burton in [4],[3] proved the main results on variational problems relative to class of rearrangements with bounded support. Burton and Nycander extended some of them to stratified rearrangements. The present investigation will focus on extension of some results on spaces of rearrangements to the stratified case.
Let $\Omega \subset \mathbb{R}^3$ be a bounded domain and $q_0 \in L^2(\Omega)$. The set of all rearrangements of q_0 on Ω is denoted by $\mathcal{F}_\Omega(q_0)$, and the closed convex hull in $L^2(\Omega)$ of $\mathcal{F}_\Omega(q_0)$ is denoted by $\Im_\Omega(q_0)$. Now we can define

$$S\mathcal{F}_\Omega(q_0) = \{q \in L^2(\Omega) : q(.,z) \in \mathcal{F}_{\Omega(z)}(q_0(.,z)) \text{ for a.e. real } z\},$$

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$$S\Im_\Omega(q_0) = \{q \in L^2(\Omega) : q(.,z) \in \Im_{\Omega(z)}(q_0(.,z)) \text{ for a.e. real } z\},$$

where $\Omega(z) := \{(x,y) : (x,y,z) \in \Omega\}$. Since $q_0(.,z) \in L^2(\Omega(z))$, definitions are well defined and we refer to elements of $S\mathcal{F}(q_0)$ as stratified rearrangements of q_0. Burton [4] proved the following theorem,

Theorem 1.1. *Let $f, g \in L^2(\Omega)$ be non-negative functions. Suppose that there is a function φ such that $f^* := \varphi(g)$ is a rearrangement of f where φ is an increasing function. Then f^* is the unique maximizer of the functional $< ., g >$ relative to $\Im_\Omega(f)$.*

In this paper, we investigate the validity of the same result to the stratified rearrangements and more other results.

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PERTURBATION OF THE WIGNER EQUATION IN INNER PRODUCT C^*-MODULES

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Abstract. Let $\mathfrak{A}$ be a C^*-subalgebra of a von Neumann algebra $\mathfrak{B}$. Let $\mathcal{M}$ and $\mathcal{N}$ be inner product modules over $\mathfrak{A}$ and $\mathfrak{B}$, respectively. Under some assumptions we show that for each mapping $f\colon \mathcal{M} \to \mathcal{N}$ satisfying

$$\|\,|\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\,\| \le \varphi(x,y) \qquad (x, y \in \mathcal{M}),$$

where φ is a control function, there exists a solution $I\colon \mathcal{M} \to \mathcal{N}$ of the Wigner equation

$$|\langle I(x), I(y)\rangle| = |\langle x, y\rangle| \qquad (x, y \in \mathcal{M})$$

such that

$$\|f(x) - I(x)\| \le \sqrt{\varphi(x,x)} \qquad (x \in \mathcal{M}).$$

1. Introduction and preliminaries

In this paper, we deal with a perturbation of the Wigner equation, establishing a link between two topics: inner product C^*-modules and functional equations.

1.1. **Inner product C^*-modules.** Let $(\mathfrak{A}, \|\cdot\|)$ be a C^*-algebra and let $\mathcal{X}$ be an algebraic right $\mathfrak{A}$-module which is a complex linear space with $(\lambda x)a = x(\lambda a) = \lambda(xa)$ for all $x \in \mathcal{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. The space $\mathcal{X}$ is called a *(right) inner product $\mathfrak{A}$-module* (*inner product C^*-module over the C^*-algebra $\mathfrak{A}$, pre-Hilbert $\mathfrak{A}$-module*) if there exists an $\mathfrak{A}$-valued inner product, i.e., a mapping $\langle\cdot,\cdot\rangle\colon \mathcal{X} \times \mathcal{X} \to \mathfrak{A}$ satisfying

(i) $\langle x, x\rangle \ge 0$ and $\langle x, x\rangle = 0$ if and only if $x = 0$,

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* Speaker.

(ii) $\langle x, \lambda y + z\rangle = \lambda\langle x, y\rangle + \langle x, z\rangle$,
(iii) $\langle x, ya\rangle = \langle x, y\rangle a$,
(iv) $\langle y, x\rangle = \langle x, y\rangle^*$,

for all $x, y, z \in \mathcal{X}$, $a \in \mathfrak{A}$, $\lambda \in \mathbb{C}$. One can show that $\|x\|_{\mathcal{X}} = \sqrt{\|\langle x, x\rangle\|}$ is a norm on $\mathcal{X}$, so $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ is a normed space. If this normed space is complete, then $\mathcal{X}$ is called a *Hilbert* $\mathfrak{A}$*-module*, or a *Hilbert* C^**-module over the* C^**-algebra* $\mathfrak{A}$. For an inner product $\mathfrak{A}$-module $\mathcal{X}$, let $\mathcal{X}^{\#}$ be the set of all bounded $\mathfrak{A}$-linear mappings from $\mathcal{X}$ into $\mathfrak{A}$, that is, the set of all bounded linear mappings $f : \mathcal{X} \to \mathfrak{A}$ such that $f(xa) = f(x)a$ for all $x \in \mathcal{X}$, $a \in \mathfrak{A}$. Every $x \in \mathcal{X}$ gives rise to a mapping $\hat{x} \in \mathcal{X}^{\#}$ defined by $\hat{x}(y) = \langle x, y\rangle$ for all $y \in \mathcal{X}$. A Hilbert module $\mathcal{X}$ is called *self-dual* if $\mathcal{X}^{\#} = \{\hat{x} : x \in \mathcal{X}\}$.

More information on inner product modules can be found e.g. in monographs [5].

1.2. **Wigner equation.** We will be considering the *Wigner equation* $|\langle I(x), I(y)\rangle| = |\langle x, y\rangle| \qquad (x, y \in \mathcal{M})$, (W), where $I\colon \mathcal{M} \to \mathcal{N}$ is a mapping between inner product modules $\mathcal{M}$ and $\mathcal{N}$ over certain C^*-algebras.

We say that two mappings $f, g\colon \mathcal{M} \to \mathcal{N}$ are *phase-equivalent* if and only if there exists a scalar valued mapping $\xi\colon \mathcal{M} \to \mathbb{C}$ such that $|\xi(x)| = 1$ and $f(x) = \xi(x)g(x)$ for all $x \in \mathcal{M}$. The equation (W) has been introduced already in 1931 by E.P. Wigner in the realm of (complex) Hilbert spaces. The classical Wigner's theorem, stating that a solution of (W) has to be phase-equivalent to a unitary or antiunitary operator, has deep applications in physics (quantum mechanics). One of the proofs of this remarkable result can be found e.g. in [4]. Recently, Wigner's result has been studied in the realm of Hilbert modules (cf. e.g. [1]). Stability of the orthogonality equation in Hilbert C^*-modules has been studied in [3]. For more information on stability of the Wigner equation in the framework of Hilbert spaces see [2].

2. Main results

Suppose that we are given a *control mapping* $\varphi\colon \mathcal{M} \times \mathcal{M} \to [0, \infty)$ satisfying, with some constant $0 < c \neq 1$, the conditions

$$\text{(2.1)}\qquad \begin{array}{ll} \text{(a)} & \lim_{n\to\infty} c^n\varphi(c^{-n}x, y) = 0,\ \lim_{n\to\infty} c^n\varphi(x, c^{-n}y) = 0 \quad (x, y \in \mathcal{M}); \\ \text{(b)} & \text{the sequence } \left(c^{2n}\varphi(c^{-n}x, c^{-n}x)\right) \text{ is bounded for any } x \in \mathcal{M}. \end{array}$$

We say that a mapping $f\colon \mathcal{M} \to \mathcal{N}$ *approximately* satisfies the Wigner equation if

$$(\mathrm{W}_\varphi) \qquad \|\, |\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\, \| \le \varphi(x, y) \qquad (x, y \in \mathcal{M}).$$

The question we would like to answer is if each solution of (W_φ) can be approximated by a solution of (W).

Let us consider the following condition on an inner product C^*-module $\mathcal{X}$.

[H] For each norm-bounded sequence (x_n) in $\mathcal{X}$, there exists a subsequence (x_{l_n}) of (x_n) and $x_0 \in \mathcal{X}$ such that

$$\|\langle x_{l_n}, y\rangle - \langle x_0, y\rangle\| \to 0 \text{ (as } n \to \infty) \qquad \text{for all } y \in \mathcal{X}.$$

Validity of [H] in Hilbert spaces follows from its reflexivity and the fact that each ball is sequentially weakly compact. It is an interesting question to characterize the class of all inner product C^*-modules in which [H] is satisfied. We however have the following result.

Proposition 2.1. *If $\mathcal{X}$ is a Hilbert C^*-module over a finite-dimensional C^*-algebra $\mathfrak{A}$, then the condition* [H] *is satisfied.*

Theorem 2.2. *Let $\mathfrak{A}$ be a C^*-subalgebra of a von Neumann algebra $\mathfrak{B}$ acting on a Hilbert space $\mathcal{H}$. Let $\mathcal{M}$ be an inner product $\mathfrak{A}$-module and let $\mathcal{N}$ be an inner product $\mathfrak{B}$-module satisfying* [H]. *Then, for each mapping $f\colon \mathcal{M} \to \mathcal{N}$ satisfying* (W_φ), *with φ satisfying* (2.1), *there exists $I\colon \mathcal{M} \to \mathcal{N}$ with the following properties:*

(i) $\langle I(x), I(x)\rangle = \langle x, x\rangle \qquad (x \in \mathcal{M})$,
(ii) *I preserves orthogonality, that is, $\langle x, y\rangle = 0$ implies $\langle I(x), I(y)\rangle = 0$,*
(iii) $\|f(x) - I(x)\| \le \sqrt{\varphi(x, x)} \qquad (x \in \mathcal{M})$.

Furthermore, there exists $h\colon \mathcal{M} \to \mathcal{N}$ such that the following decomposition holds:

$$f(x) = h(x) + I(x), \quad \langle h(x), I(x)\rangle = 0 \quad \text{and} \quad \|h(x)\| \le \sqrt{\varphi(x, x)} \qquad (x \in \mathcal{M}).$$

If $\mathfrak{B}$ is abelian, then I can be chosen as a solution of (W).

Corollary 2.3. *Let $\mathfrak{A}$ be a C^*-subalgebra of a von Neumann algebra $\mathfrak{B}$. Let $\mathcal{M}$ be an inner product $\mathfrak{A}$-module and let $\mathcal{N}$ be an inner product $\mathfrak{B}$-module satisfying* [H]. *Let either $p, q \ge 1$ or $p, q \le 1$ (but not $p = q = 1$) and $\varepsilon > 0$. Then, for each mapping $f\colon \mathcal{M} \to \mathcal{N}$ satisfying*

$$\|\,|\langle f(x), f(y)\rangle| - |\langle x, y\rangle|\,\| \le \varepsilon \|x\|^p \|y\|^q \qquad (x, y \in \mathcal{M}),$$

there exists $I\colon \mathcal{M} \to \mathcal{N}$ preserving orthogonality such that $\langle I(x), I(x)\rangle = \langle x, x\rangle$ and $\|f(x) - I(x)\| \le \sqrt{\varepsilon}\,\|x\|^{(p+q)/2}$ for all $x \in \mathcal{M}$. Moreover, if $\mathfrak{B}$ is abelian, then I can be chosen as a solution of (W).

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AMENABILITY OF SEMIGROUP ALGEBRAS AND THEIR SECOND DUALS

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ABSTRACT. We recall the definitions of amenable and weakly amenable Banach algebras, and give equivalent formulations. We define the semigroup algebra of a semigroup, and explain when these algebras are amenable. We explain that their second duals are never amenable when the semigroup is infinite. We also state that the second dual of a group algebra $L^1(G)$, for a locally compact group G, is weakly amenable if and only if G is finite.

1. INTRODUCTION AND PRELIMINARIES

The notion of an amenable Banach algebra was first formulated by the late Barry Johnson in 1972 [9]; independently, Alexander Helemskii of Moscow came to an equivalent definition. There is a considerable discussion of these notions, and some history, in the monograph [2] and in the text [10] of Helemskii. This has proved to be a very fruitful idea in the study of the general theory of Banach algebras and of the many examples of such algebras; it is surprising that a definition that at first sight seems rather technical should in so many cases delineate significant classes of Banach algebras. Indeed 'amenability' should be seen as some substitute for 'finiteness' in a wide class of mathematical structures. It is thus a natural challenge to determine the specification of the amenable Banach algebras within the classical classes.

The notion of 'weakly amenable Banach algebras' (see below for the definition) was introduced by Bade, Curtis, and Dales in [1], and this too has proved of considerable, but lesser, importance in the theory. In recent years, many variants of 'amenability' have been considered. These include 'Connes-amenability' (see [13]), which takes into account the dual structure of some Banach algebras, and various notions of 'approximate amenability', as developed by Ghahramani and Loy [8, 5].

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Here are some examples of successful theories about amenable Banach algebras, and of some open questions. Most proofs can be found in [2].

• Let G be a locally compact group. Then the original memoir of Johnson [9] established that a group algebra $L^1(G)$ is amenable if and only if the group G is itself amenable (and this is the reason for the name 'amenable'); indeed, in this case, $L^1(G)$ is 1-amenable (see below). Johnson also proved that $L^1(G)$ is always weakly amenable. Later work of Dales, Ghahramani, and Helemskii [3], resolved the question when the measure algebra $M(G)$ of G is amenable and when it is weakly amenable.

• The question which C^*-algebras are amenable has been a major theme within the mathematics of the last century; there are now many characterizations of amenable C^*-algebras, including the fact that a C^*-algebra is amenable if and only if it is nuclear. These results are asscoiated with such famous names as Alain Connes, Ed. Effros, and Uffe Haagerup. For example, all commutative C^*-algebras are amenable, and the 'small' C^*-algebra $\mathcal{K}(H)$ of all compact operators on a Hilbert space H are amenable, but the 'large' C^*-algebra $\mathcal{B}(H)$ of all bounded linear operators on H is not amenable (for H infinite-dimensional). See [13] for much more on this.

• A uniform algebra A contained in $C(X)$, for a compact space X, is amenable if and only if $A = C(X)$. However it is not known whether or not the only uniform algebras A which are weakly amenable are equal to $C(X)$.

• Let E be an infinite-dimensional Banach space. It is conjectured that $\mathcal{B}(E)$ is never amenable. This has been noted above for the case where $E = \ell^{\,2}$, and the same result has been proved by Charles Read for the space $E = \ell^{\,1}$ by a different method [12]. Different proofs of the same result have been given by Pisier and by Osawa. However it is remarkable that, even the case where $E = \ell^{\,p}$ for $1 < p < 2$ is open.

• Charles Read [11] has a remarkable example (see [2]) of a commutative, radical Banach algebra that is amenable.

2. Main results

Our talk considers the questions when a semigroup algebra $\ell^{\,1}(S)$ is amenable, and when its second dual $(\ell^{\,1}(S)'', \Box)$ is amenable or weakly amenable. The intuition is that S should be 'restricted' and 'small' if the algebra $\ell^{\,1}(S)$ be amenable, and that the second dual $(\ell^{\,1}(S)'', \Box)$ is such a 'large' algebra that it should never be amenable. Our results substantiate this intuition.

All the results in this lecture are taken from the memoir [4] of Dales, Lau, and Strauss; this memoir contains many more theorems about different aspects of the algebras $\ell^{\,1}(S)$ and $(\ell^{\,1}(S)'', \Box)$.

2.1. **Modules and derivations.** Let A be a Banach algebra, and let $(E, \|\cdot\|)$ be a Banach space which is also an A-bimodule. Then E is a *Banach A-bimodule* if

$$\|a \cdot x\| \leq \|a\|\,\|x\|\,, \quad \|x \cdot a\| \leq \|a\|\,\|x\| \quad (a \in A,\, x \in E)\,.$$

For example, $E = A$ is a Banach A-bimodule with module operations the product m_A. The *dual* of a Banach A-bimodule E is the Banach space E' for the operations

$$\langle x,\, a \cdot \lambda \rangle = \langle x \cdot a,\, \lambda \rangle, \quad \langle x,\, \lambda \cdot a \rangle = \langle a \cdot x,\, \lambda \rangle \quad (a \in A,\, x \in E,\, \lambda \in E')\,.$$

For example, A' and A'' are Banach A-bimodules. The Banach space $(A \widehat{\otimes} A, \|\cdot\|_\pi)$ is a Banach A-bimodule for continuous maps satisfying the conditions that $a \cdot (b \otimes c) = ab \otimes c$ and $(b \otimes c) \cdot a = b \otimes ca$ for $a, b, c \in A$. There is a map $\pi : a \otimes b \mapsto ab, \ \ A \otimes A \to A$; this is a continuous A-bimodule map.

Definition 2.1. Let A be an algebra, and let E be an A-bimodule. A *derivation* is a linear map $D : A \to E$ such that

$$D(ab) = Da \cdot b + a \cdot Db \quad (a, b \in A) .$$

For example, $\delta_x : a \to a \cdot x - x \cdot a$ is a derivation; these are called *inner* derivations. A *point derivation* at a character φ of A is a linear functional $d : A \to \mathbb{C}$ such that

$$d(ab) = \varphi(a)d(b) + \varphi(b)d(a) \quad (a, b \in A).$$

Definition 2.2. Let A be a Banach algebra. Then A is *amenable* if each continuous derivation from A into each dual Banach A-bimodule is inner, and *weakly amenable* if each continuous derivation from A into the dual Banach A-bimodule A' is inner.

In practice it is usually easier to work from the following intrinsic characterization of amenable Banach algebras.

Definition 2.3. A *bounded approximate diagonal* for A is a bounded net $(u_\alpha) \in A \widehat{\otimes} A$such that $\pi(u_\alpha) = e_A$ and $\|a \cdot u_\alpha - u_\alpha \cdot a\|_\pi \to 0 \quad (a \in A)$.

Theorem 2.4. (Barry Johnson) *A unital Banach algebra is amenable if and only if A has a bounded approximate diagonal. In this case the* **constant of amenability** *is the minimum C such that there is a bounded approximate diagonal (u_α) with* $\sup \|u_\alpha\|_\pi \leq C$, *and now A is C-***amenable**.

2.2. **Semigroups.** A *semigroup* is a non-empty set with an associative binary operation. Let S be a semigroup. Then $p \in S$ is an *idempotent* if $p^2 = p$; the set of these is $E(S)$.

Special semigroups, called *Rees semigroups with a zero*, were introduced by Rees in 1941, and play a prominent role in the theory of semigroups. These are the semigroups $S^o = \mathcal{M}^o(G, P, m, n)$. Here $m, n \in \mathbb{N}$, G is a group, and $P = (a_{ij}) \in \mathbb{M}_{n,m}(G^o)$. For $x \in G$, $i \in \mathbb{N}_m$, and $j \in \mathbb{N}_n$, let $(x)_{ij}$ be the element of $\mathbb{M}_{m,n}(G^o)$ with x in the $(i, j)^{\text{th.}}$ place and o elsewhere. The semigroup S consists of the matrices $(x)_{ij}$. Multiplication is given by: $(x)_{ij}(y)_{k\ell} = (xa_{jk}y)_{i\ell}$ for $x, y \in G$, $i, k \in \mathbb{N}_m$, $j, \ell \in \mathbb{N}_n$. We add a zero o to obtain S^o. The matrix P is the *sandwich* matrix.

A semigroup S is *regular* if, for each $s \in S$, there exists $t \in S$ with $sts = s$. An *ideal* I in S is a subset such that $SI \cup IS \subset I$; a minimum ideal in S (if it exists) is called $K(S)$. The structure theorem of Rees is as follows.

Theorem 2.5. *Let S be an infinite, regular semigroup for which $E(S)$ is finite. Then $K(S)$ exists and is a group, and S has a principal series*

$$S = I_1 \supset I_2 \supset \cdots \supset I_{m-1} \supset I_m = K(S)$$

such that each quotient I_j/I_{j+1} is a Rees semigroup with a zero.

Definition 2.6. Let S be a semigroup, and let $A = \ell^1(S)$. Set

$$\left(\sum \alpha_r \delta_r\right) \star \left(\sum \beta_s \delta_s\right) = \sum \left\{ \sum_{rs=t} \alpha_r \beta_s \delta_t : t \in S \right\} .$$

for $\sum \alpha_r \delta_r, \sum \beta_s \delta_s \in A$. Then $(A, \star)$ is the *semigroup algebra* of S.

Note that, if S has an identity e, then so does A, and $\|\delta_e\|_1 = 1$. But A can have an identity with norm $\neq 1$. Is the norm of the identity always a rational number?

Our theorem is as follows. It extends results of Duncan and Paterson [6] and of Esslamzadeh [7].

Theorem 2.7. *Suppose that $\ell^1(S)$ is amenable as a Banach algebra. Then $K(S)$ exists and is an amenable group, and each of the semigroups $\mathcal{M}^o(G, P, m, n)$ that arise in the principal series are such that $m = n$, G is an amenable group, and P is invertible when regarded as a matrix in $\mathbb{M}_n(\ell^1(G))$. Further the converse is true.*

We can also gain some information about the constant of amenability of a semigroup algebra.

Theorem 2.8. *Let S be a semigroup with $\ell^1(S)$ amenable. Then the following are equivalent:* (a) $C_S = 1$*;* (b) $E_S = 1$*;* (c) $C_S < 5$*;* (d) *S is a group. Further* 5 *is the best-possible constant in* (c)*.*

So numbers in the range $(1, 5)$ are 'forbidden values' for C_S. We can obtain numbers of the form $4k + 1$ for C_S: are there any more possibilities?

This leads to the following further results, which answer open questions. The second dual of a Banach algbera A with the first Arens product (see [2]) is denoted by $(A'', \Box)$.

Theorem 2.9. *Suppose that $(\ell^1(S)'', \Box)$ is amenable as a Banach algebra. Then S is finite.*

Theorem 2.10. *Let G be a locally compact group. Then the following are equivalent:* (a) *the group G is infinite;* (b) *there is a non-zero, continuous point derivation at the discrete augmentation character of $(L^1(G)'', \Box)$;* (c) *$(L^1(G)'', \Box)$ is not weakly amenable.*

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REFLEXIVE R_DUAL SEQUENCES AND FRAMES

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ABSTRACT. Dual sequences and dual frames play a fundamental role for analyzing frame theory and Gabor systems. In this article we present reflexive R_dual sequences and reflexive frames with respect to two orthonormal basis in a separable Hilbert space $\mathcal{H}$. Also we give some equivalent conditions for reflexivity of R_dual frames.

1. INTRODUCTION AND PRELIMINARIES

A sequence $(f_i)_{i\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$ is called a frame if there are $0 < A \leq B < \infty$ such that

$$A \parallel f \parallel^2 \leq \sum_{i\in\mathbb{N}} \mid \langle f, f_i\rangle \mid^2 \leq B \parallel f \parallel^2 \qquad (f \in \mathcal{H})$$

and it is called a frame sequence if it is a frame only for its closed linear span, and the frame operator $S : \mathcal{H} \to \mathcal{H}$ is defined by

$$Sf = \sum_{i\in\mathbb{N}} \langle f, f_i\rangle f_i.$$

Two frames $(f_i)_{i\in\mathbb{N}}$ and $(g_i)_{i\in\mathbb{N}}$ for $\mathcal{H}$ are equivalent, if there exists an invertible operator $U : \mathcal{H} \to \mathcal{H}$ such that $Uf_i = g_i$ for all $i \in \mathbb{N}$.

A sequence $(f_i)_{i\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$ is called a Riesz basis if there are $0 < A \leq B < \infty$ such that

$$A \parallel c \parallel^2 \leq \left\| \sum_{i\in\mathbb{N}} c_i f_i \right\|^2 \leq B \parallel c \parallel^2 \qquad (c = (c_i)_{i\in\mathbb{N}}).$$

A sequence $(f_i)_{i\in\mathbb{N}}$ in $\mathcal{H}$ is complete, if the span of $(f_i)_{i\in\mathbb{N}}$ is dense in $\mathcal{H}$. A sequence $(f_i)_{i\in\mathbb{N}}$ is called ω-independent if for any sequences of scalars $c = (c_i)_{i\in\mathbb{N}}$, and $\sum_i c_i f_i = 0$, it follows $c = 0$.

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* Speaker.

For a sequence $f = (f_i)_{i\in\mathbb{N}}$ in a separable Hilbert space $\mathcal{H}$, we define R-dual sequences $\omega(f)$ and $v(f)$ respect to two orthonormal sequences $(e_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ in $\mathcal{H}$.

Definition 1.1. Let$(e_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ be two orthonormal sequences in a separable Hilbert space $\mathcal{H}$. Let f =$(f_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{i\in\mathbb{N}} \mid \langle f_i, e_j\rangle \mid^2 < \infty$ for each $j \in \mathbb{N}$, and let

$$\omega_j(f) = \sum_{i\in\mathbb{N}} \langle f_i, e_j\rangle h_i \qquad (j \in \mathbb{N}). \tag{1.1}$$

Then $\omega(f) :=(\omega_j(f))_{j\in\mathbb{N}}$ is called the R-dual sequence for the sequence $(f_i)_{i\in\mathbb{N}}$ with respect to $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$.

2. Reflexive R_Dual Sequences and Frames

If $\sum_{i\in\mathbb{N}} \mid \langle f_i, h_j\rangle \mid^2 < \infty$, we define $v(f) = (v_j(f))_{j\in\mathbb{N}}$ as

$$v_j(f) = \sum_{i\in\mathbb{N}} \langle f_i, h_j\rangle e_i \qquad (j \in \mathbb{N}). \tag{2.1}$$

Definition 2.1. In a separable Hilbert space $\mathcal{H}$, the sequence $(f_i)_{i\in\mathbb{N}}$ is called a reflexive $R_$dual sequence with respect to $(e_j)_{j\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ if $v(f) = \omega(f)$.
The reflexive R_dual frame is a frame that is reflexive R_dual.

Theorem 2.2. *Let $(e_i)_{i\in\mathbb{N}}$ and $(h_i)_{i\in\mathbb{N}}$ be two orthonormal sequences in a separable Hilbert space $\mathcal{H}$. Let f =$(f_i)_{i\in\mathbb{N}}$ be a sequence in $\mathcal{H}$ such that $\sum_{i\in\mathbb{N}} \mid \langle f_i, e_j\rangle \mid^2 < \infty$ and $\sum_{i\in\mathbb{N}} \mid \langle f_i, h_j\rangle \mid^2 < \infty$. If B is the matrix of $(h_i)_{i\in\mathbb{N}}$ with respect to the orthonormal basis $(e_j)_{j\in\mathbb{N}}$, C is the matrix of $(e_j)_{j\in\mathbb{N}}$ with respect to $(h_i)_{i\in\mathbb{N}}$, $A = (\langle f_i, e_j\rangle)_{i\in\mathbb{N},j\in\mathbb{N}}$ and $A' = (\langle f_i, h_j\rangle)_{i\in\mathbb{N},j\in\mathbb{N}}$, then the following statements hold:*
(1) $\omega(v(f)) = v(\omega(f)) = f$.
(2) A is invertible if and only if A' is invertible.

Proof. To show (1) we have

$$\begin{aligned}(\omega(v(f)))_i &= \sum_j \langle v_j(f), e_i\rangle h_j = \sum_j \langle f_i, h_j\rangle h_j = f_i\\ (v(\omega(f)))_i &= \sum_j \langle \omega_j(f), h_i\rangle e_j = \sum_j \langle f_i, e_j\rangle e_j = f_i.\end{aligned}$$

So we have $\omega(v(f)) = v(\omega(f)) = f$.
Since we have

$$\begin{aligned}(AB^*)_{ik} &= \sum_{j=1}^{\infty} \langle f_i, e_j\rangle \bar{b}_{kj}\\ &= \sum_{j=1}^{\infty} \langle f_i, e_j\rangle \overline{\langle h_k, e_j\rangle}\\ &= \langle f_i, \sum_{j=1}^{\infty} \langle h_k, e_j\rangle e_j\rangle\\ &= \langle f_i, h_k\rangle = (A')_{ik}\end{aligned}$$

and since $BB^* = I$ so $B^* = B^{-1} = C$ and hence $A' = AB^* = AC$, thus (2) holds. □

Theorem 2.3. *Let $(e_i)_{i\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$, $f = (f_i)_{i\in\mathbb{N}}$ be as in Theorem 2.2. Then $(f_i)_{i\in\mathbb{N}}$ is an R-dual sequence for $\omega(f)$ with respect to $(h_i)_{i\in\mathbb{N}}$ and $(e_j)_{j\in\mathbb{N}}$ and is an R-dual sequence for $v(f)$ with respect to $(e_j)_{j\in\mathbb{N}}$and $(h_i)_{i\in\mathbb{N}}$.*

Theorem 2.4. *Let $(e_j)_{j\in\mathbb{N}}$, $(h_i)_{i\in\mathbb{N}}$, $(f_i)_{i\in\mathbb{N}} \subset H$, and B, C, A and A' be as in Theorem 2.2. Then the following conditions are equivalent:*
(1) $\omega_j(f) = v_j(f)$
(2) $\bar{B}AB^* = A$
(3) $A^*C^tAS^{-1}C = I$.

Theorem 2.5. *$\omega(f)$ is a frame sequence if and only if $v(f)$ is a frame sequence.*

In [1] we show that for all $(a_j)_{j\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$,

$$\| \sum_{j=1}^{\infty} a_j\omega_j(f) \|^2 = \sum_{i=1}^{\infty} | \langle \Phi, f_i \rangle |^2$$

and

$$\| \sum_{i=1}^{\infty} b_i f_i \|^2 = \sum_{j=1}^{\infty} | \langle g, \omega_j(S_e^{-\frac{1}{2}} f) \rangle |^2$$

where $\Phi = \sum_j \bar{a}_j e_j$ and $g = \sum_i \bar{b}_i h_i$.

Theorem 2.6. *For all $(a_j)_{j\in\mathbb{N}}, (b_i)_{i\in\mathbb{N}} \in \ell^2(\mathbb{N})$,*

$$\| \sum_{j=1}^{\infty} b_j v_j(f) \|^2 = \sum_{i=1}^{\infty} | \langle g, f_i \rangle |^2$$

and

$$\| \sum_{i=1}^{\infty} a_i f_i \|^2 = \sum_{j=1}^{\infty} | \langle \Phi, v_j(f) \rangle |^2,$$

where $\Phi = \sum_j \bar{a}_j e_j$ and $g = \sum_i \bar{b}_i h_i$.

Theorem 2.7. *$\omega(f)$ is complete if and only if $v(f)$ is complete.*

Corollary 2.8. *$\omega(f)$ is a frame if and only if $v(f)$ is a frame.*

Theorem 2.9. *$\omega(f)$ is a Riesz basis if and only if $v(f)$ is a Riesz basis.*

Theorem 2.10. *$\omega(f)$ is ω-independent if and only if $v(f)$ is ω-independent.*

Theorem 2.11. *$span(\omega_j(f)) = span(\omega_j(g))$ if and only if $span(v_j(f)) = span(v_j(g))$.*

Theorem 2.12. *Let $(e_j)_{j\in\mathbb{N}}$,$(h_i)_{i\in\mathbb{N}}$,$(f_i)_{i\in\mathbb{N}}$,$\omega(f)$ and $v(f)$ be defined as in Theorem 2.2. If $\omega(f)$ and $v(f)$ are frames then $\omega(f)$ is equivalent to $v(f)$ if and only if $\ker A = \ker A'$.*

Proof. Let $a = (a_j)_{j\in\mathbb{N}}$ be any sequence of scalers. First we have

$$\begin{aligned} \sum_{j=1}^{\infty} a_j v_j(f) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_j \langle f_i, h_j \rangle e_i \\ &= \sum_{i=1}^{\infty} (A'a)_i e_i. \end{aligned}$$

This implies,

$$\sum_{j=1}^{\infty} a_j v_j(f) = 0 \Leftrightarrow A'a = 0.$$

Secondly, we have

$$\begin{aligned}\sum_{j=1}^{\infty} a_j \omega_j(f) &= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} a_j \langle f_i, e_j\rangle h_i \\ &= \sum_{i=1}^{\infty} (Aa)_i h_i.\end{aligned}$$

Hence,

$$\sum_{j=1}^{\infty} a_j \omega_j(f) = 0 \Leftrightarrow Aa = 0.$$

So $\ker(A) = \ker(A')$ if and only if for all sequences of scalars $(a_j)_{j\in\mathbb{N}}$, we have $\sum_j a_j v_j(f) = 0$ if and only if $\sum_j a_j \omega_j(f) = 0$, and this is true if and only if $(v_j(f))_{j\in\mathbb{N}}$ is equivalent to $(\omega_j(f))_{j\in\mathbb{N}}$. □

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ON THE NUMERICAL STABILITY OF CHAOTIC SYNCHRONIZATION USING NONLINEAR COUPLING FUNCTION

GHOLAM HOSSEIN ERJAEE

ABSTRACT. A nonlinear coupling has been used for synchronization of some chaotic systems. The difference evolutional equation between coupled systems, determined via the linear approximation, can be used for stability analysis of the synchronization between drive and response systems. According to the stability criteria the coupled chaotic systems are asymptotically synchronized, if all eigenvalues of their matrices found in this linear approximation have negative real parts. Obviously, there is no synchronization, if at least one of these eigenvalues has positive real part. Nevertheless, in this paper we have considered some cases on which there is at least one zero eigenvalue for the matrix in the linear approximation. In such cases, there will be a synchronization-like behavior between coupled chaotic systems, if all other eigenvalues have negative real part.

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APPROXIMATE AMENABILITY AND WEAKLY APPROXIMATE AMENABILITY OF THE SECOND DUALS OF BANACH ALGEBRAS

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Abstract. In this paper we study the approximately amenability and the weakly approximate amenability of second dual $\mathcal{A}^{**}$ of the Banach algebra $\mathcal{A}$. Indeed we show that the weakly approximate amenability of second dual $\mathcal{A}^{**}$ implies the weakly approximate amenability of $\mathcal{A}$ if one of the following assertions holds (i) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$, (ii) $\mathcal{A}$ is a dual Banach algebra, (iii) $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is weakly compact.

1. Introduction and preliminaries

Let $\mathcal{A}$ be a Banach algebra and X a Banach $\mathcal{A}$-module, that is X is a Banach space and an $\mathcal{A}$-module such that the module operations $(a, x) \longmapsto ax$ and $(a, x) \longmapsto xa$ from $\mathcal{A} \times X$ into X are jointly continuous. Then X^* is also a Banach $\mathcal{A}$-module if we define

$$\langle x, ax^* \rangle = \langle xa, x^* \rangle, \quad \langle x, x^*a \rangle = \langle ax, x^* \rangle \qquad (a \in \mathcal{A},\ x \in X,\ x^* \in X^*).$$

A derivation from $\mathcal{A}$ into X is a continuous linear operator D such that

$$D(ab) = a \cdot D(b) + D(a) \cdot b \qquad (a, b \in \mathcal{A}).$$

We define $\delta_x(a) = a \cdot x - x \cdot a$; for each $x \in X$ and $a \in \mathcal{A}$. δ_x is a derivation from $\mathcal{A}$ into X, which is called an inner derivation. A derivation $D : \mathcal{A} \longrightarrow X$ is called approximately inner if there exists a net (x_α) such that for each $a \in \mathcal{A}$, we have $D(a) = \lim_\alpha \delta_{x_\alpha}(a)$. A Banach algebra $\mathcal{A}$ is amenable if every derivation

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* Speaker.

from $\mathcal{A}$ into every dual $\mathcal{A}-$bimodule X^* is inner i.e. $H^1(\mathcal{A}, X^*) = \{0\}$. This definition was introduced by B. E. Johnson in 1972 (for more details see [1]). A Banach algebra $\mathcal{A}$ is weakly amenable if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is inner i.e. $H^1(\mathcal{A}, \mathcal{A}^*) = \{0\}$. Bade, Curtis and Dales in 1987 have introduced the concept of weak amenability for commutative Banach algebras (see [1]). A Banach algebra $\mathcal{A}$ is approximately amenable if every derivation from $\mathcal{A}$ into X^* is approximately inner, for each $\mathcal{A}-$bimodule X . A Banach algebra $\mathcal{A}$ is weakly approximately amenable if every derivation from $\mathcal{A}$ into $\mathcal{A}^*$ is approximately inner. Ghahramani and Loy [4] have introduced the concept of approximate amenability and weakly approximate amenability .

Throughout this paper, the first and the second Arens products are respectively denoted by $\square$ and $\Diamond$. These products can be defined by

$$F\square G = w^* \lim_i \lim_j \hat{f}_i \hat{g}_j \qquad and \qquad F\Diamond G = w^* \lim_j \lim_i \hat{f}_i \hat{g}_j,$$

where (f_i) and (g_j) are nets of elements of $\mathcal{A}$ such that $\hat{f}_i \longrightarrow F$ and $\hat{g}_j \longrightarrow G$ in w^*-topology (see [1]).

2. Main results

Theorem 2.1. *Let $\mathcal{A}$ be a Banach algebra. Suppose that $(\mathcal{A}^{**}, \square)$ is weakly approximately amenable. If one of the following assertions holds:*
(i) $\mathcal{A}$ is a left ideal in $\mathcal{A}^{**}$;
(ii) $\mathcal{A}$ is a dual algebra;
(iii) $\mathcal{A}$ is Arens regular and every derivation from $\mathcal{A}$ into its dual is weakly compact;
then $\mathcal{A}$ is weakly approximately amenable.

Proof. Let $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ be a derivation. If one of the conditions (i), (ii) or (iii) holds, then D has an extension $\bar{D} : \mathcal{A}^{**} \longrightarrow (\mathcal{A}^{**})^*$ such that $\bar{D}$ is a derivation (see[2], [3] and [5]). Since $\mathcal{A}^{**}$ is weakly approximately amenable there exists a net $(\psi_i) \subseteq A^{***}$ such that for every $F \in \mathcal{A}^{**}, (\Lambda \circ \bar{D})(F) = \lim_i(F.\psi_i - \psi_i.F)$. Put ϕ_i to be the restriction of ψ_i to $\mathcal{A}$. Then for every $a \in \mathcal{A}$, we have $D(a) = \lim_i(a.\phi_i - \phi_i.a)$. Thus $\mathcal{A}$ is weakly approximately amenable. $\square$

Theorem 2.2. *Let $\mathcal{A}$ be a weakly approximately amenable Banach algebra . then $\mathcal{A}^2$ is dense in $\mathcal{A}$.*

Proof. Let $\overline{\mathcal{A}^2} \neq \mathcal{A}$. Then by Hahn Banach theorem there exist $f_0 \in \mathcal{A}^*$ such that $f_0 \neq 0$ and $\mathcal{A}^2 \subseteq ker f$. Define $D : \mathcal{A} \longrightarrow \mathcal{A}^*$ by $D(a) = f_0(a)f_0$. It is easily checked that D is a derivation but is not approximately inner. For if D is approximately inner then there exists a net $(\psi_i) \in \mathcal{A}$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_i(a.\psi_i - \psi_i.a)$. Thus for every $a \in \mathcal{A}$, $(f_0(a))^2 = D(a)(a) = \lim_i(\psi_i(a^2) - \psi_i(a^2)) = 0$. Hence $f_0(a) = 0$. Thus D is not approximately inner. $\square$

Theorem 2.3. *Let $\mathcal{B}$ be a closed subalgebra of $\mathcal{A}^{**}$ such that $\widehat{\mathcal{A}} \subseteq \mathcal{B}$. If $\mathcal{B}$ is approximately amenable, then $\mathcal{A}$ is approximately amenable.*

Proof. By $[5, Lemma 1.7]$ there is a continuous linear mapping $\psi : (\mathcal{A}^\sharp)^{**}\hat{\otimes}(\mathcal{A}^\sharp)^{**} \longrightarrow (\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{**}$ such that for every $a, b, c \in \mathcal{A}^\sharp$ and $m \in (\mathcal{A}^\sharp)^{**}\hat{\otimes}(\mathcal{A}^\sharp)^{**}$,

(1) $\psi(\hat{a}\hat{\otimes}\hat{b}) = \widehat{(a\hat{\otimes}b)}$,

(2) $\psi(m).c = \psi(m.c)$,
(3) $c.\psi(m) = \psi(c.m)$,
(4) $\pi^{**}_{\mathcal{A}^\sharp}(\psi(m)) = \pi_{(\mathcal{A}^\sharp)^{**}}(m)$.

Let $J : (\mathcal{B}^\sharp\hat{\otimes}\mathcal{B}^\sharp) \longrightarrow (\mathcal{A}^\sharp)^{**}\hat{\otimes}(\mathcal{A}^\sharp)^{**}$ be defined as $J(a\hat{\otimes}b) = a\hat{\otimes}b$. Since $\mathcal{B}$ is approximately amenable, then there exists a net $(N_\alpha) \subseteq (\mathcal{B}^\sharp\hat{\otimes}\mathcal{B}^\sharp)^{**}$ such that for every $x \in \mathcal{B}^\sharp$ we have $\lim_\alpha(x.N_\alpha - N_\alpha.x) = 0$ and $\pi^{**}_{\mathcal{B}^\sharp}(N_\alpha) = e$ for every α. Let $\theta : (\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^* \longrightarrow (\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{***}$ be canonical embedding. Now consider the net $(M_\alpha) = (\theta^* \circ \psi^{**} \circ J^{**}(N_\alpha)) \subseteq (\mathcal{A}^\sharp\hat{\otimes}\mathcal{A}^\sharp)^{**}$. First, for every $x \in \mathcal{A}^\sharp$,

$$\lim_\alpha(x.M_\alpha - M_\alpha.x) = \lim_\alpha((\theta^* \circ \psi^{**} \circ J^{**})(x.N_\alpha - N_\alpha.x)) = 0.$$

Second, for every $\alpha \in I$ there exists $(N^\beta) \subseteq (\mathcal{B}^\sharp\hat{\otimes}\mathcal{B}^\sharp)$ such that $w^* - \lim_\beta \widehat{N^\beta} = N_\alpha$. Thus

$$\begin{aligned}
\pi^{**}_{\mathcal{A}^\sharp}(M_\alpha) &= \pi^{**}_{\mathcal{A}^\sharp}(\theta^* \circ \psi^{**} \circ J^{**}(N_\alpha)) = w^* - \lim \pi^{**}_{\mathcal{A}^\sharp}(\theta^* \circ \psi^{**}(\widehat{J(N^\beta)})) \\
&= w^* - \lim \pi^{**}_{\mathcal{A}^\sharp}(\theta^*(\psi(\widehat{J(N^\beta)}))) = w^* - \lim \pi^{**}_{\mathcal{A}^\sharp}(\psi(J(N^\beta))) \\
&= w^* - \lim \pi_{(\mathcal{A}^\sharp)^{**}}(J(N^\beta)) = \pi^{**}_{(\mathcal{B}^\sharp)}(N_\alpha) = e.
\end{aligned}$$

So $\mathcal{A}$ is approximately amenable. □

Corollary 2.4. *Let $\mathcal{A}$ be a Banach algebra such that $\mathcal{A}^{**}$ is approximately amenable then $\mathcal{A}$ is approximately amenable.*

For a Banach algebra $\mathcal{A}$, let $\mathcal{A}^{op}$ be the Banach algebra with underlying Banach space same as $\mathcal{A}$ and with product $\circ$ given by $a \circ b = ba$.

Lemma 2.5. *Let $\mathcal{A}$ be a Banach algebra. Then $\mathcal{A}$ is approximately amenable (weakly approximately amenable) if and only if $\mathcal{A}^{op}$ is approximately amenable (weakly approximately amenable).*

Lemma 2.6. *Let $\mathcal{A}$, $\mathcal{B}$ be Banach algebras and let $\theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a continuous isomorphism. If $\mathcal{A}$ is approximately amenable (weakly approximately amenable), then $\mathcal{B}$ is approximately amenable (weakly approximately amenable).*

By applying the above Lemma we can prove the following theorem.

Theorem 2.7. *Let $\mathcal{A}$ be a Banach algebra with a continuous anti-isomorphism $\varphi : \mathcal{A} \longrightarrow \mathcal{A}$. Then $(\mathcal{A}^{**}, \Box)$ is approximately amenable if and only if $(\mathcal{A}^{**}, \Diamond)$ is approximately amenable.*

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UPPER BOUND FOR MATRIX OPERATOR HAUSDORFF

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Abstract. In this paper, we considere the problem of finding an upper bound for Hausdorff matrix operator from sequence spaces $l_p(v)(ord(v,p))$ into $l_p(w)(ord(w,p))$, and deduce an extension of Hardy inequality on weighted sequence spaces.

1. Introduction

Let $w = (w_n)$ be a sequence with positive entries. For $p \geq 1$, we define the weighted sequence space $l_p(w)$ as follows:

$$l_p(w) = \{ \ (x_n) \ : \ \sum_{n=1}^{\infty} w_n |x_n|^p < \infty \ \},$$

with norm, $\|.\|_{p,w}$, which is defined as follows:

$$\|x\|_{w,p} = (\sum_{n=1}^{\infty} w_n |x_n|^p)^{1/p}.$$

Also, if $w = (w_n)$ is a decreasing sequence of positive numbers such that $\lim_{n\to\infty} w_n = 0$ and $\sum_{n=1}^{\infty} w_n = \infty$, then the Lorentz sequence space $d(w,p)$ is defined as follows:

$$d(w,p) = \{ \ (x_n) \ : \ \sum_{n=1}^{\infty} w_n x_n^{*p} < \infty \ \},$$

where (x_n^*) is the decreasing rearrangement of $(|x_n|)$. In fact $d(w,p)$ is the space of null sequences x for which x^* is in $l_p(w)$, with norm $\|x\|_{d(w,p)} = \|x^*\|_{w,p}$.
Our objective in this paper is to give a generalization of some results obtained by

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* Speaker.

Bennett [1], Hardy [2] and Jameson and Lashkaripour [4], for Hausdorff matrix operators on the weighed sequence space. We deduce an upper bound for certain matrix operators Cesaro, Euler, Holder and Gamma.
We consider the Hausdorff matrix operator $H(\mu) = (h_{j,k})$, such that

$$h_{j,k} = \begin{cases} \begin{pmatrix} j \\ k \end{pmatrix} \int_0^1 \theta^{k-1}(1-\theta)^{j-k} d\mu(\theta) & \text{if} \quad 1 \le k \le j \\ 0 & \text{if} \quad k > j. \end{cases}$$

where μ is a probability measure on $[0, 1]$.
The Hausdorff matrix is contained in the famous classes of matrices. These classes are as follow:
(i) the choice $d\mu(\theta) = \alpha(1-\theta)^{\alpha-1}d\theta$ gives the Cesaro matrix of order α;
(ii) the choice $d\mu(\theta) =$ *point evaluation at* $\theta = \alpha$ gives the Euler matrix of order α;
(iii) the choice $d\mu(\theta) = \frac{|\log\theta|^{\alpha-1}}{\Gamma(\alpha)}d\theta$ gives the Holder matrix of order α;
(iv) the choice $d\mu(\theta) = \alpha\theta^{\alpha-1}d\theta$ gives the Gamma matrix of order α.

Now consider the operator A defined by $Ax = y$, where $y_i = \sum_{i=1}^{\infty} a_{i,j}x_j$. We write $\|A\|_{v,w,p}$ for the norm of A as an operator from $l_p(v)$ into $l_p(w)$, and $\|A\|_{w,p}$ for the norm of A as an operator from $l_p(w)$ into itself, and $\|A\|_p$ for the norm of A as an operator from l_p into itself, and $\|A\|_{d(w,p)}$ for the norm of A as an operator from $d(w,p)$ into itself.
The following conditions are what we need to convert statements for $l_p(w)$ to ones for $d(w,p)$. We assume throughout that
(1)For all i, j, $\quad a_{i,j} \ge 0$.
(2) For all subsets M, N of natural numbers having m, n elements respectively, we have $\sum_{i\in M}\sum_{j\in N} a_{i,j} \le \sum_{i=1}^{m}\sum_{j=1}^{n} a_{i,j}$.
(3) $\sum_{i=1}^{\infty} w_i a_{i,1}$ is convergent.
Condition (1) implies that $|A(x)| \le A(|x|)$ and hence the non-negative sequences are sufficient for examine norm of A.
Proposition 1. Let $p \ge 1$ and $A = (a_{i,j})$ be an operator with conditions (1) and (2), then $\|A(x)\|_{d(w,p)} \le \|A(x^*)\|_{d(w,p)}$, for all non-negative elements x of $d(w,p)$. Hence decreasing, non-negative elements are sufficient to determine the norm of A.
Condition (3) ensures that at least finite sequence are mapped into $d(w, 1)$.
Proposition 2.([4], Lemma 1). Let $p \ge 1$ and $A = (a_{i,j})$ be an operator with non-negative entries. Also, let A maps $d(w,p)$ into itself. If for $x \in d(w,p)$, we set $Ax = y$ with $y_i = \sum_{j=1}^{\infty} a_{i,j}x_j$, then the following conditions are equivalent:
(1) $y_1 \ge y_2 \ge \cdots \ge 0$ when $x_1 \ge x_2 \ge \cdots \ge 0$.
(2) $r_{i,n} = \sum_{j=1}^{n} a_{i,j}$ decreases with respect to i for each n.

In the following statement, we assume (v_n) and (w_n) are non-negative decreasing sequences with $v_1 = 1$.

Theorem 2. Let $H(\mu)$ be the Hausdorff matrix operator and $p > 1$. Then the Hausdorff matrix operator maps $l_p(v)$ into $l_p(w)$, and

$$(\inf \frac{w_n}{v_n})^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta)) \le \|H\|_{v,w,p} \le (\sup \frac{w_n}{v_n})^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Therefore the Hausdorff matrix operator maps $l_p(w)$ into itself, and

$$\|H\|_{w,p} = \int_0^1 \theta^{-1/p} d\mu(\theta).$$

Proof. Let x be a non-negative sequence. Then since (w_n) is decreasing, applying Theorem 216 of [3], we have

$$\begin{aligned}
\|Hx\|_{w,p}^p &= \sum_{j=1}^{\infty} w_j (\sum_{k=1}^{j} \begin{pmatrix} j \\ k \end{pmatrix} (\int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta)) x_k)^p \\
&\leq \sum_{j=1}^{\infty} (\sum_{k=1}^{j} \begin{pmatrix} j \\ k \end{pmatrix} (\int_0^1 \theta^k (1-\theta)^{j-k} d\mu(\theta)) w_k^{1/p} x_k)^p \\
&\leq (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \sum_{j=1}^{\infty} w_j x_j^p \\
&= (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \sum_{j=1}^{\infty} \frac{w_j}{v_j} v_j x_j^p \\
&\leq \sup \frac{w_j}{v_j} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \|x\|_{v,p}^p.
\end{aligned}$$

Hence

$$\|Hx\|_{w,p} \leq (\sup \frac{w_n}{v_n})^{1/p} (\int_0^1 \theta^{-1/p} d\mu(\theta)) \|x\|_{v,p},$$

and so

$$\|H\|_{v,w,p} \leq (\sup \frac{w_n}{v_n})^{1/p} \int_0^1 \theta^{-1/p} d\mu(\theta).$$

It remains to prove the left-hand inequality. We take

$$0 \leq \delta < \frac{1}{p}, \qquad x_n = (n+1)^{-\frac{1}{p}-\delta}$$

and any positive ϵ, where $0 < \epsilon < 1$; and choose α, and N such that:

$$\begin{aligned}
(1+\frac{1}{\alpha})^{-2/p} &> 1-\epsilon, \\
\int_{\alpha/n}^1 \theta^{-1/p} d\mu(\theta) &> (1-\epsilon) \int_0^1 \theta^{-1/p} d\mu(\theta) \qquad (n \geq N), \\
\sum_{n=N}^{\infty} w_n x_n^p &> (1-\epsilon) \sum_{n=1}^{\infty} w_n x_n^p.
\end{aligned}$$

Since $(x_n) \in l_p$, and $0 < v_n \leq 1$, we deduce that $(x_n) \in l_p(v)$. Also, we have

$$(Hx)_n = \sum_{m=1}^{n} \begin{pmatrix} n \\ m \end{pmatrix} (\int_0^1 \theta^m (1-\theta)^{n-m} d\mu(\theta)) x_m \geq (1-\epsilon)^2 x_n \int_0^1 \theta^{-1/p} d\mu(\theta)$$

for all $n \geq N$ and so

$$w_n^{1/p} (Hx)_n \geq (1-\epsilon)^2 w_n^{1/p} x_n \int_0^1 \theta^{-1/p} d\mu(\theta) \qquad (n \geq N).$$

Hence

$$\begin{aligned} \|Hx\|_{w,p}^p &\geq \sum_{n=N}^{\infty} w_n (Hx)_n^p \\ &\geq (1-\epsilon)^{2p} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \sum_{n=N}^{\infty} w_n x_n^p \\ &\geq (1-\epsilon)^{2p+1} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \sum_{n=1}^{\infty} w_n x_n^p \\ &= (1-\epsilon)^{2p+1} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \sum_{n=1}^{\infty} \frac{w_n}{v_n} v_n x_n^p \\ &\geq \inf \frac{w_n}{v_n} (1-\epsilon)^{2p+1} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \|x\|_{v,p}^p. \end{aligned}$$

Since ϵ is arbitrary, if $\epsilon \longrightarrow 0$, we have

$$\|Hx\|_{w,p}^p \geq \inf \frac{w_n}{v_n} (\int_0^1 \theta^{-1/p} d\mu(\theta))^p \|x\|_{p,v}^p,$$

and this completes the proof of the statement.

Corollary 1. Let $p > 1$ and $p^* = \frac{p}{p-1}$, then Cesaro, Holder, Gamma and Euler operators map $l_p(w)$ into $l_p(w)$. Also, we have:

$$\begin{aligned} \|C(\alpha)\|_{w,p} &= \frac{\Gamma(\alpha+1)\Gamma(1/p^*)}{\Gamma(\alpha+\frac{1}{p^*})} \qquad (\alpha > 0); \\ \|H(\alpha)\|_{w,p} &= \frac{1}{\Gamma(\alpha)} \int_0^1 \theta^{-\frac{1}{p}} |\log\theta|^{\alpha-1} d\theta \qquad (\alpha > 0); \\ \|\Gamma(\alpha)\|_{w,p} &= \frac{\alpha p}{\alpha p - 1} \qquad (\alpha p > 1); \\ \|E(\alpha)\|_{w,p} &= \alpha^{-1/p} \qquad (0 < \alpha < 1). \end{aligned}$$

Corollary 2. If x and w are non-negative sequences and w is decreasing, then

$$\sum_{n=1}^{\infty} w_n (\frac{1}{n} \sum_{i=1}^{n} x_i)^p \leq p^{*p} (\sum_{n=1}^{\infty} w_n x_n^p).$$

Proof. Apply Corollary 1, for Cesaro operator with $\alpha = 1$.

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AMENABLE BANACH ALGEBRA AND SCALAR-TYPE SPECTRAL OPERATORS

MOHAMMAD BAGHER GHAEMI *[1] AND ALI REZA HOSSEINIUN[2]

ABSTRACT. Willis showed that, if T is a compact operator on Hilbert space and $\mathcal{A}_T$ is amenable, then T is scalar-type spectral operator. The properties of Hilbert spaces are used in essential way at several points in the Willis proof and it does not work for Banach spaces. By an Example we will show that there is a compact operator T on a Banach Space X such that $\mathcal{A}_T$ is amenable, and T is not scalar type Spectral. This answers a question raised by wellis.

1. Introduction and preliminaries

Amenability is a finiteness condition for Banach algebras which in many cases has strong consequence for the algebra. There are no general theorems about the structure of T, other than the obvious remark, since $\mathcal{A}_T$ has an approximate identity, T cannot be nilpotent. It is not known, for example, whether an amenable commutative Banach algebra can be radical.Hence we cannot yet answer the questions which are usually the first to be asked about operators. However it is possible to say more by considering operators satisfying further conditions. Willis in [9] showed that, if T is a compact operator on Hilbert space and $\mathcal{A}_T$ is amenable, then T is similar to a normal operator. The properties of Hilbert spaces are used in essential way at several points in the Willis proof and it does not work for Banach spaces.By an Example we will show that there is a compact operator T on a Banach Space X such that $\mathcal{A}_T$ is amenable and T is not scalar-type Spectral. The class of scalar -type spectral operators on a Banach space was introduced by Dunford [3] as a natural analogue of the normal operators on Hilbert spaces. They can be characterized by their possession of a weakly compact functional calculus for continuous functions on the spectrum ([8], Theorem) or ([7], Corollary 1).

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2. Scalar-type Operators

Definition 2.1. *An operator T in $L(X)$ is called a prespectral operator of class Γ if the following conditions are satisfied:*

(1) *There is a spectral measure $E(\cdot)$ of class $(\Sigma_p,\ \Gamma)$ with values in $L(X)$ such that*
$$TE(\delta) = E(\delta)\,T \qquad (\delta \in \Sigma_p)$$
where Σ_p denotes the σ- algebra of Borel subsets of the complex plane.

(2)
$$\sigma(T|E(\delta)X) \subseteq \overline{\delta} \qquad (\delta \in \Sigma_p).$$

The spectral measure $E(\cdot)$ is called a resolution of the identity of class Γ for T.

Definition 2.2. *Let S be a prespectral operator on X with resolution of the identity $E(\cdot)$ of class Γ such that*
$$S = \int_{\sigma(T)} \lambda E(d\lambda).$$
Then S is called a scalar-type operator of class Γ.

Definition 2.3. *An operator $T \in L(X)$ is a spectral operator if there is a spectral measure $E(\cdot)$ defined on Σ_p with values in $L(X)$ such that*

(1) *$E(\cdot)$ is countably additive on Σ_p in the strong operator topology,*
(2) *$TE(\tau) = E(\tau)\,T \qquad (\tau \in \Sigma_p)$,*
(3) *$\sigma(T|E(T)X) \subseteq \overline{\tau} \qquad (\tau \in \Sigma_p)$.*

Remark 2.4. *The operator $S \in L(X)$ is spectral if and only if it is prespectral of class X'.*

Theorem 2.5. *Let $S \in L(X)$. Then the following conditions are equivalent:*

(1) *$S' \in L(X')$ is a scalar-type of class X,*
(2) *$S \in L(X)$ is strongly normal-equivalent,*
(3) *there exist a compact subset Ω of $\mathbb{C}$ and a norm continuous representation $\Theta : C(\Omega) \mapsto X$ such that $\Theta(z \mapsto z) = S$, $\Theta(z \mapsto 1) = I$.*

Proof. [4], Theorem 2.4. □

Theorem 2.6. *Let X be a Banach space which does not contain a subspace isomorphic to c_0. If $S' \in L(X')$ is scalar-type of class X, then $S \in L(X)$ is scalar-type spectral.*

Proof. [4], Corollary 2.5. □

Definition 2.7. *The operator $T \in L(H)$ is similar to a normal operator if there is a self-adjoint operator U, invertible in $L(H)$, such that the operator UTU^{-1} is normal.*

Theorem 2.8. *Let S be scalar spectral operator on H. There is a self-adjoint operator B, invertible in $L(H)$ such that the operator BSB^{-1} is normal.*

Proof. [2], Theorem 8.3. □

3. Amenable Banach algebras

A Banach algebra $\mathcal{A}$ is amenable if for every Banach $\mathcal{A}$- bimodul $\mathcal{X}$, every derivation $D : A \to \mathcal{X}'$, is inner. Here a Banach $\mathcal{A}$-bimodul is an $\mathcal{A}$-bimodule $\mathcal{X}$, which is a Banach space such that the bimodule maps $(a, x) \to x.a$ and $(a, x) \to a.x$ are jointly continuous from $\mathcal{A} \times \mathcal{X}$ to $\mathcal{X}$. If XIf $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule, then the dual space, $\mathcal{X}'$, is also, with the $\mathcal{A}$-actions defined by $\langle x, a.x'\rangle = \langle a.x, x'\rangle$ and $\langle x, x'.a\rangle = \langle a.x, x'\rangle$, $(a \in \mathcal{A};\ x \in \mathcal{X};\ \textit{and}\ x' \in \mathcal{X}')$. A derivation is a linear map, $D : \mathcal{A} \to \mathcal{X}$,such that $D(ab) = a.D(b) + D(a).b$, $(a, b \in \mathcal{A})$. For each $x \in \mathcal{X}$, the map $D_x : a \mapsto a.x - x.a$, is a derivation and such a derivation are called inner.These notations are explained in more detail in [1] and [6].

The class of amenable Banach algebras is stable under operations which construct a new algebra from old one such stability properties will be needed is the following.

Theorem 3.1. *If $\mathcal{A}$ is amenable Banach algebra and $\Theta : \mathcal{A} \to \mathcal{B}$ is a homomorphism with $\Theta(\mathcal{A})$ dense in $\mathcal{B}$, then $\mathcal{B}$ is amenable.*

Proof. [?], Proposition 5.3. □

Theorem 3.2. *If $T \in L(X)$ is an operator with dual of scalar-type and $\sigma(T) \subseteq \mathbb{R}$ then the Banach algebra generated by T is amenable.*

Proof. Suppose $T' : X' \to X'$ is scalar-type of class X by Theorem 2.5 T has a $C(\sigma(T))$-functional calculus. Therefore the Banach algebra generated T is amenable. □

The following Corollaries is immediate from Theorems (2.5 and 3.2).

Corollary 3.3. *If $S \in L(X)$ is strongly normal-equivalent and $\sigma(T) \subseteq \mathbb{R}$, then the Banach algebra generated by T is amenable.*

Corollary 3.4. *If $S \in L(X)$ is scalar-type prespectral and $\sigma(T) \subset \mathbb{R}$ then the Banach algebra generated by T is amenable.*

Corollary 3.5. *If $T \in L(X)$ is a well bounded operator with decomposition of identity of bounded variation the The Banach algebra generated by T is amenable.*

Proof. T is a well-bounded operator with decomposition of the identity of bounded variation now by ([2], Theorem 16.15. T' is scalar-type spectral of class X and now the result follows from Theorem 3.2. □

Theorem 3.6. *Let T be a compact operator on Hilbert space, $\mathcal{H}$, and suppose that $\mathcal{A}_T$ is amenable. Then T is similar to a normal operator (scalar-type spectral operator).*

The Theorem 3.6 is due to Willis. The Theorems 3.6, is another aspect of amenability of $\mathcal{A}_T$ as diagonability of T.

The next example shows in the case of Banach spaces we can not have such a strong result as Theorem 3.6 and at most the adjoint of operator T is scalar-type of class X.

Example 3.7. *Let X be the Banach space of all convergent sequences $w = \{\beta_n\}$ of complex numbers under the norm*

$$\|w\| = \sup |\beta_n|.$$

The paring of X' with l_1 given by

$$\langle w, f\rangle = \lambda_1\beta_n + \sum_{n=1}^{\infty}\beta_n\lambda_{n+1},$$

where $f = \{\lambda_n\} \in l_1$, induces an isometric isomorphism of l_1 onto X'. Define $T \in L(X)$, by

$$T\{\beta_n\} = \{-1/n\beta_+ n\}.$$

Then T is a compact operator and for any complex polynomial p we have

$$\|p(T)\| \leq \sup\{|p(t)| : -1 \leq t \leq 0\}.$$

Now by ([2]*, Theorem 16.15) T' is scalar-type of class X and by Theorem 3.2 the Banach algebra generated by T is amenable, and T is not scalar type spectral.*

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OPERATORS COMMUTING WITH TRANSLATIONS AND CONJUGATIONS ON GROUP ALGEBRAS

ALI GHAFFARI*

ABSTRACT. We discusses recent developments in the theory of continuous linear maps on group algebras. Some of the basic concepts in this area went back to Wendel who studied multipliers in the operators theory. I will give a survey of recent results and applications to different subjects such as translation operators, convolution operators and conjugation operators on group algebras. Further, some results about the modulus of right multipliers are investigated.

1. INTRODUCTION AND PRELIMINARIES

Let G be a locally compact group with left Haar measure and let $L^p(G)$, $1 \leq p \leq \infty$, be the complex Lebesgue spaces associated with it. If $x \in G$, then L_x will denote the translation operator which defines a linear isometry of $L^1(G)$ onto $L^1(G)$.

The first Arens product on $L^1(G)^{**}$ is obtained by making in turn the definitions

$$\langle f\phi, \psi \rangle = \langle f, \phi * \psi \rangle, \quad \langle Hf, \phi \rangle = \langle H, f\phi \rangle, \quad \langle FH, f \rangle = \langle F, Hf \rangle.$$

If we regard $L^1(G)$ as embedded canonically in $L^1(G)^{**}$ we find $\phi\psi = \phi * \psi$. Again starting from above, we find $f\phi = \hat{\phi} * f$ (where $\hat{\phi}(x) = \Delta(x^{-1})\phi(x^{-1})$).

2. MAIN RESULTS

Let G be a locally compact Abelian group, let X and Y be topological linear spaces of functions defined on G. We will study the set of all continuous linear maps from X to Y which commute with the translations (convolutions), i.e., $T(L_x f) = L_x T(f)$ for all $f \in X$ and $x \in G$ (, i.e., $T(\phi * f) = \phi * T(f)$ for all $f \in X$ and $\phi \in L^1(G)$).

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$M(X,Y)$ is the set of all bounded linear maps $T : X \to Y$ commuting with the translation operators.

The main result on the multiplier of $L^1(G)$ is contained in the following theorem.

Theorem 2.1. (*Wendel*) *Let G be a locally compact Abelian group and suppose $T : L^1(G) \to L^1(G)$ is a continuous linear map. Then the following are equivalent:*
(1) T commutes with the translation operators, that is, $L_x T = T L_x$ for each $x \in G$. $T(\mu * \nu) = T(\mu) * \nu$ for each $\mu, \nu \in L^1(G)$.
(2) There exists a unique measure $\mu \in M(G)$ such that $T(\nu) = \nu * \mu$ for each $\nu \in L^1(G)$.

Theorem 2.1 shows that T commutes with translations if and only if T commutes with convolutions. It is easy to see that if T is compact, then G is a compact group and $\mu \in L^1(G)$.

Remark 2.2. Let G be a locally compact Abelian group. Then the space of multipliers for $L^1(G)$ is isometrically isomorphic to $M(G)$.

Theorem 2.3. (*Larsen*) *Let G be a locally compact Abelian group and suppose $T : L^1(G) \to L^p(G)$ is a bounded linear map. Then the following are equivalent:*
(1) $T \in M(L^1(G), L^p(G))$.
(2) There exists a unique μ such that $T(\nu) = \mu * \nu$ for each $\nu \in L^1(G)$, where $\mu \in M(G)$ if $p = 1$ and $\mu \in L^p(G)$ if $1 < p \leq \infty$.

Moreover, the correspondence between T and μ defines an isometric linear isomorphism from $M(L^1(G), L^p(G))$ onto $M(G)$ if $p = 1$ and onto $L^p(G)$ if $1 < p \leq \infty$.

For a locally compact group G, let $Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ denote all bounded linear operators $T : L^\infty(G) \to L^\infty(G)$ commuting with the convolution operators L_ϕ, $\phi \in L^1(G)$, where $L_\phi(f) = \phi * f$, $f \in L^\infty(G)$, i.e., $T(\phi * f) = \phi * T(f)$ (see [2]). Then, as a well known fact, $Hom_{L^1(G)}(L^\infty(G), L^\infty(G)) \subseteq M(L^\infty(G), L^\infty(G))$.

We know that for $F \in L^1(G)^{**}$, $T_F : f \mapsto Ff$ is a continuous linear map from $L^\infty(G)$ to itself. If we take $\phi \in L^1(G)$ then, for all $\psi \in L^1(G)$,

$$\langle F(f\phi), \psi\rangle = \langle F, (f\phi)\psi\rangle = \langle F, f(\phi\psi)\rangle = \langle Ff, \phi\psi\rangle = \langle (Ff)\phi, \psi\rangle.$$

Thus each F provides an element of $Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$. Baker, Lau and Pym [1] proved that the correspondence between T and F defines an isometric isomorphism from $Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ onto $LUC(G)^*$, where $LUC(G)$ denote the space of all $f \in C_b(G)$ such that the mapping $x \to {}_x f$ from G to $C_b(G)$ is continuous.

Example 2.4. a) Let G be a locally compact Abelian group. We know that $L_0^\infty(G)$ is the space of bounded measurable functions on G which vanish at infinity. As known, for any $F \in L_0^\infty(G)^*$ and $f \in L_0^\infty(G)$, $Ff \in L_0^\infty(G)$. In this case, the first Arens multiplication is well defined on $L_0^\infty(G)^*$, and $L_0^\infty(G)^*$ is a Banach algebra. It is easy to see that $T(Ff) = FT(f)$.

b) Let A be a Banach algebra and let T be a bounded linear operator from A^* into A^* which is weak*-weak* continuous and satisfy $T(af) = aT(f)$ for all $a \in A$, $f \in A^*$. Then $T(Ff) = FT(f)$ for all $F \in A^{**}$ and $f \in A^*$.

c) Let G be a locally compact Abelian group. Let T be a compact bounded linear operator of $L^\infty(G)$ to itself and satisfy $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$, $f \in L^\infty(G)$. Then G is compact and every $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$ is weak*-weak* continuous.

Theorem 2.5. *Let G be a locally compact Abelian group. Then the following conditions are equivalent:*
(1) G is a compact group.
(2) for every bounded linear operator $T \in Hom_{L^1(G)}(L^\infty(G), L^\infty(G))$, $T(Ff) = FT(f)$ for all $F \in L^1(G)^{**}$ and $f \in L^\infty(G)$.

If $y \in G$ and f is a complex function on G, we define ${}_yf_y(x) = f(y^{-1}xy)$, for $x \in G$. For every $\phi \in L^1(G)$ and $f \in L^p(G)$ $(1 \leq p < \infty)$ the $\star$-convolution $\phi \star f$,

$$\phi \star f(x) = \int \phi(y) f(y^{-1}xy) \Delta(y)^{\frac{1}{p}} dy \quad (x \in G),$$

exists and represents an element of $L^p(G)$ of norm $\|\phi \star f\|_p \leq \|\phi\|_1 \|f\|_p$. For every $\phi \in L^1(G)$ and $f \in L^\infty(G)$, we define $\phi \star f(x) = \int \phi(y) f(y^{-1}xy) dy$.

Theorem 2.6. *Let G be a locally compact group and $1 \leq p < \infty$. If T is a bounded linear operator from $L^p(G)$ into $L^p(G)$, then the following statements are equivalent:*
(1) T commutes with the conjugation operators, that is, $T({}_yf_y) = {}_yT(f)_y$ for all $f \in L^p(G)$ and $y \in G$.
(2) T commutes with convolution, i.e., $T(\phi \star f) = \phi \star T(f)$ for all $f \in L^p(G)$ and $\phi \in L^1(G)$.

For a Banach lattice X and an operator T on X, the modulus $|T|$ of T is defined by $|T|(x) = sup\{|T(y)|; |y| \leq x\}$ for all $x \geq 0$, provided that the supremum exists [2]. If G is a locally compact group, then for $1 \leq p \leq \infty$, the space $L^p(G)$ is a complete Banach lattice.

Theorem 2.7. (*Ghahramani and Lau*) *Suppose that T is a left multiplier on $L^1(G)$. Then $|T|$ is a left multiplier. Furthermore, if $T = \lambda_\mu$ $(\mu \in M(G))$, then $|T| = \lambda_{|\mu|}$. Moreover $|\lambda^*_\mu| = \lambda^*_{|\mu|}$.*

Theorem 2.8. *Suppose that T is a bounded linear operator on $L^\infty(G)$ such that $T({}_yf_y) = {}_yT(f)_y$ for any $y \in G$ and $f \in L^\infty(G)$. Then $|T|({}_yf_y) = {}_y|T|(f)_y$ for any $y \in G$ and $f \in L^\infty(G)$.*

For every bounded approximate identity (e_α) of $L^1(G)$ bounded by 1, if weak*-lim $e_\alpha = E$, then E is a right identity of $L^1(G)^{**}$ with $\|E\| = 1$ and there exists an isometric embedding $\Gamma_E : M(G) \to L^1(G)^{**}$ given by $\Gamma_E(\mu)$ =weak*-lim $e_\alpha * \mu$.

Theorem 2.9. *If $\mu \in M(G)$, and $\rho_\mu : L^1(G) \to L^1(G)$ is the operator of multiplication by μ on right, then the following statements hold:*
(1) $|\rho_\mu{}^*| = \rho_{|\mu|}{}^*$.
(2) If $n = \Gamma_E(\mu)$, then $|T_n| = T_{|n|}$.
(3) If $n \in LUC(G)^\perp$ and $n \neq 0$, then $|T_n| \neq T_{|n|}$, where $LUC(G)^\perp = \{F \in L^1(G)^{**}; \langle F, f\rangle = 0, f \in LUC(G)\}$.

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SOME PROPERTIES OF STRONG CHAIN RECURRENT SETS

FATEMEH GHANE

Abstract. The notion of strong chain recurrency first introduced by Easton which was stronger than chain recurrency. In this talk, we obtain some results on strong chain recurrency. We show that strong chain recurrent components are closed and invariant, and in the generic case, the strong chain recurrent set is equal to chain recurrent set. Moreover, generically, each chain transitive set is a strong chain transitive.

1. Introduction and preliminaries

Interesting dynamics are always associated with some sort of orbit recurrence.Various notions of recurrence have been considered in dynamics such as recurrent points and nonwandering points. Conley and Bowen considered a weaker notion of recurrence. They introduced the notion of chains or pseudo orbits(see[4] and [3]).

The Chain recurrency has remarkable connections to the structure of topological attractors. In 1977, Easton[5] introduced a stronger version of chain recurrency, called it strong chain recurrency. He obtained a relation between strong chain transitivity and Lipschitz ergodicity. We know that, if the dynamical systems preserve a finite Borel measure, by a consequence of Poincare recurrence theorem, the chain recurrent set is all of the ambient space. In this case, it may be justified to consider those isolated invariant sets on which the dynamical system is strong chain transitive to be important. So this is a motivation for studying the strong chain transitive sets. Here, let X be a compact metric homogeneous space with metric d and $\mathcal{H}(X)$ be the space of all homeomorphisms on X with the usual c°-topology.

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Recall that a subset is residual if it contains a countable intersection of open and dense sets. We say that a property is generic, if it holds on a residual subset. For given $\epsilon > 0$, an ϵ-chain is a sequence of points $x_0, \ldots, x_n \in X$ such that $d(f(x_{k-1}), x_k) < \epsilon$, for $k = 1, \ldots, n$. An invariant set Λ is said to be chain-transitive, if for each $p, q \in \Lambda$ and $\epsilon > 0$, there exists an ϵ-chain $x_0, \ldots, x_n$ such that $x_0 = p$ and $x_n = q$. Define Λ to be strong chain transitive if given $p, q \in \Lambda$ and $\epsilon > 0$, there exists an ϵ-chain $x_0, \ldots, x_n$ from p to q such that $\sum_{k=1}^{n} d(f(x_{k-1}), x_k) < \epsilon$. Such a chain is called a ϵ-strong chain. If for each $\epsilon > 0$, there exists an ϵ-chain from x to itself, then x is said to be chain-recurrent. However, if there is an ϵ-strong chain from x to itself, then x is called a strong chain-recurrent point, we denote the sets of all chain-recurrent points and strong chain-recurrent points of f by $CR(f)$ and $SCR(f)$, respectively. Now, we define two binary relations which are equivalence relations and their equivalence classes are the main object of this article. For each $x, y \in CR(f)$, we say that $x \sim_C y$ if and only if for any $\epsilon > 0$, there is an ϵ-chain of f from x to y and conversely from y to x. Similarly, if $x, y \in SCR(f)$ then $x \sim_S y$ if and only if there exists an ϵ-strong chain from x to y and from y to x, for each $\epsilon > 0$. Clearly $\sim_C$ and $\sim_S$ define equivalence relations on $CR(f)$ and $SCR(f)$, respectively. The equivalence classes of $\sim_C$ and $\sim_S$ are called chain recurrence classes and strong chain recurrence classes or chain classes and strong chain classes, briefly.

1.1. **Main results.** Here, we present the following results.

Proposition 1.1. *Let$f \in \mathcal{H}(X)$. If S is a strong chain component for f, then S is a closed and invariant subset of X. In particular, if $X \in S$ then $\omega(x, f) \subset S$.*

Proposition 1.2. *There exists a residual set R_1 of $\mathcal{H}(X)$, such that for each $f \in \mathcal{R}_1$, $\Omega(f) \subseteq SCR(f) \subseteq CR(f)$.*

Proposition 1.3. *If f satisfies the shadowing property, then each chain transitive set is strong chain transitive.*

Proposition 1.4. *Let k be a positive integer. Then there exists a residual set R_2 of $\mathcal{H}(X)$ such that for each $f \in \mathcal{R}_2$, $SCR(f^k) = SCR(f)$.*

Proposition 1.5. *Let k be a positive integer. Then for each $f \in \mathcal{H}(X)$, $SCR(f^k) \subset SCR(f)$. Moreover, if S_0 is a strong chain component of f^k, then $S = \bigcup_{i=0}^{k-1} f^i(B_0)$ is the strong chain component of f containing S_0.*

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POLYNOMIAL AND RATIONAL APPROXIMATION IN BANACH FUNCTION ALGEBRAS

TAHER GHASEMI HONARY*

Abstract. Let K and X be compact plane sets such that $K \subseteq X$. We define

$$P(X,K) = \{f \in C(X) : f|_K \in P(K)\}$$
$$R(X,K) = \{f \in C(X) : f|_K \in R(K)\}$$

where $P(K)$ and $R(K)$ are uniform closures of polynomials and rational functions with poles off K, respectively. Let $S_0(A)$ denote the set of peak points of the Banach function algebra A on X. Let S and T be compact subsets of the compact plane set X. We first show that if the symmetric difference $S + T$ has planar measure zero then $R(X,S) = R(X,T)$, which implies the Hartogs - Rosenthal theorem. Then we show that the following properties are equivalent:
(i) $R(X,S) = R(X,T)$.
(ii) $S \setminus T \subseteq S_0(R(X,S))$ and $T \setminus S \subseteq S_0(R(X,T))$.
(iii) $R(K) = C(K)$ for every compact subset $K \subseteq X \setminus (S \cap T)$.
Moreover, $P(X,S) = P(X,T)$ if and only if $S \setminus T \subseteq S_0(P(X,S))$ and $T \setminus S \subseteq S_0(P(X,T))$.

Finally, we show that some of the above properties are satisfied for the extended Lipschitz algebras.

1. Introduction and preliminaries

The algebra of all continuous complex-valued functions on the compact Hausdorff space X is denoted by $C(X)$. The subalgebra $A \subseteq C(X)$ is called a Banach function algebra on X if A separates the points of X, contains the constants and is complete under an algebra norm. If the norm of the Banach function algebra A is the uniform

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norm then it is called a uniform algebra. We denote the maximal ideal space of A by Φ_A.

Let K, S, T and X be compact subsets of $\mathbb{C}^n$ such that $K, S, T \subseteq X$, and let $P_0(K)$, $R_0(K)$ be the algebras of all polynomials and rational functions on K with poles off K, respectively. The uniform closures of $P_0(K)$ and $R_0(K)$ are denoted by $P(K)$ and $R(K)$, respectively, which are uniform algebras on K. The polynomial convex hull of K is denoted by $\hat{K}$. The set K is called polynomially convex if $\hat{K} = K$. Let m denote the planar measure. A theorem due to Hartogs-Rosenthal asserts that $R(K) = C(K)$ if the compact plane set K has planar measure zero [1, II.8.4] or [2]. It is also known that $R(K) = C(K)$ if and only if every point of K is a peak point for $R(K)$ [5; 5.3.8], and $P(K) = R(K)$ if and only if K is polynomially convex. Also a theorem due to Vitushkin gives criteria for $R(K) = C(K)$ [1, VIII.5.1] or [6]. In this work we extend the above results to more general algebras of polynomials and rational functions in uniform algebras as well as Lipschitz algebras. However, some of these results are appeared in [4]. For another extension of Hartogs-Rosenthal see [3].

If we take $P_0(X, K) = \{f \in C(X) : f|_K \in P_0(K)\}$ and $R_0(X, K) = \{f \in C(X) : f|_K \in R_0(K)\}$ then it is easy to see that $P(X, K) = \{f \in C(X) : f|_K \in P(K)\}$ and $R(X, K) = \{f \in C(X) : f|_K \in R(K)\}$ are, in fact, the uniform closures of $P_0(X, K)$, and $R_0(X, K)$, respectively. Note that if K is finite then $P_0(X, K) = R_0(X, K) = C(X)$ and so $P(X, K) = R(X, K) = C(X)$. Hence, we may assume that K is infinite.

It is easy to show that $P(X, K)$ and $R(X, K)$ are uniform algebras on X. Moreover, $P_0(X, K) = P_0(X)$, $R_0(X, K) = R_0(X)$, $P(X, K) = P(X)$ and $R(X, K) = R(X)$ if $K = X$.

Let A be a Banach function algebra on X. A point $p \in X$ is called a peak point for A if there exists $f \in A$ such that $f(p) = 1$ and $|f(x)| < 1$ for every $x \in X$ different from p. The set of all peak points for A is denoted by $S_0(A)$.

2. Polynomial and rational approximation in uniform algebras

Throughout this section we always assume that K, S, T and X are compact plane sets such that $K, S, T \subseteq X$.

Lemma 2.1. *If $A = \{f \in C(X) : f|_K = 0\}$, then $C_0(X \setminus K) = A|_{X \setminus K}$.*

Lemma 2.2. *Let μ be a regular complex Borel measure on X. If U is an open set in $\mathbb{C}$ such that for almost all $z \in U$, $\int_X \frac{d\mu(\zeta)}{\zeta - z} = 0$, then $\mu = 0$ on $U \cap X$.*

Theorem 2.3. *If $m(S + T) = 0$ then $R(X, S) = R(X, T)$.*

Corollary 2.4. *If $m(K) = 0$ then $R(X, K) = C(X)$. In particular, if $m(X) = 0$ then $R(X) = C(X)$, the Hartogs-Rosenthal theorem.*

Corollary 2.5. *If $m(X) = m(K)$ then $R(X, K) = R(X)$.*

Theorem 2.6. *$R(X, S) = R(X, T)$ if and only if $S \setminus T \subseteq S_0(R(X, S))$ and $T \setminus S \subseteq S_0(R(X, T))$.*

Corollary 2.7. *$R(X, K) = R(X)$ if and only if $X \setminus K \subseteq S_0(R(X))$.*

Corollary 2.8. *$R(X, K) = C(X)$ if and only if $K \subseteq S_0(R(X, K))$.*

Theorem 2.9. *$P(X,S) = P(X,T)$ if and only if $S \setminus T \subseteq S_0(P(X,S))$ and $T \setminus S \subseteq S_0(P(X,T))$.*

Theorem 2.10. *The following assertions are equivalent:*
$(i) R(X,S) = R(X,T)$.
(ii) For every compact subset $K \subseteq X \setminus (S \cap T)$, $R(K) = C(K)$.
(iii) For every compact subset $K \subseteq X \setminus (S \cap T)$, and for every open set D, $\gamma(D \setminus K) = \gamma(D)$, where γ is the analytic capacity.
(iv) For every compact subset $K \subseteq X \setminus (S \cap T)$, and for almost all $z \in K$ (with respect to the planar measure) we have $\limsup\limits_{r \to 0^+} \frac{\gamma(\Delta(z;r) \setminus K)}{r} > 0$, where $\Delta(z;r)$ is the closed disk with centre z and radius r.

3. Polynomial and Rational Approximation in Extended Lipschitz Algebras

Definition 3.1. Let (X,d) be a compact metric space, K be a compact subset of X and $0 < \alpha \leq 1$. We define $Lip(X,K,\alpha) = \{f \in C(X) : p_{\alpha,K}(f) < \infty\}$, where

$$p_{\alpha,K}(f) = \sup\{\frac{|f(x) - f(y)|}{d^\alpha(x,y)} : x, y \in K, x \neq y\}.$$

The subalgebra $lip(X,K,\alpha)$ is defined as the set of all $f \in C(X)$ such that $\lim \frac{|f(x)-f(y)|}{d^\alpha(x,y)} = 0$, as $d(x,y) \to 0$, while $x, y \in K$.

If we define the norm $\|.\|$ on $Lip(X,K,\alpha)$ by $\|.\| = \|.\|_X + p_{\alpha,K}(.)$ then $Lip(X,K,\alpha)$ and $lip(X,K,\alpha)$ are Banach function algebras on X. Moreover, they are natural, i.e. their maximal ideal spaces coincide with X.

Definition 3.2. Let K,X be compact plane sets such that $K \subseteq X$. For $0 < \alpha \leq 1$, let $Lip_P(X,K,\alpha)$ and $Lip_R(X,K,\alpha)$ denote the subalgebras of $Lip(X,K,\alpha)$, which are generated by $P_0(X,K)$ and $R_0(X,K)$, respectively. The subalgebras $lip_P(X,K,\alpha)$ and $lip_R(X,K,\alpha)$ are defined similarly for $0 < \alpha < 1$.

Note that $lip_P(X,K,\alpha) = Lip_P(X,K,\alpha)$ and $lip_R(X,K,\alpha) = Lip_R(X,K,\alpha)$ for $0 < \alpha < 1$. Clearly, $Lip_P(X,K,\alpha)$ and $Lip_R(X,K,\alpha)$ are uniformly dense in $P(X,K)$, and $R(X,K)$, respectively. Moreover, for $0 < \alpha < 1$, $lip_R(X,K,\alpha) = lip(X,K,\alpha)$ if and only if $lip_R(K,\alpha) = lip(K,\alpha)$. Hence by [3; Theorem 2], $lip_R(X,K,\alpha) = lip(X,K,\alpha)$ if $m(K) = 0$.

Theorem 3.3. *If $m(K) = m(X)$ then $Lip_R(X,K,\alpha)$ is the closure of $R_0(X)$ in $Lip(X,K,\alpha)$.*

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MINIMAL AND MAXIMAL DESCRIPTION FOR THE REAL INTERPOLATION METHOD N-TUPLE OF QUASI-BANACH CASE

ZAHRA GHORBANI[1]* AND MOHAMMAD SADEGH MODARRES[2]

ABSTRACT. We give a minimal and maximal description in the sense of Aronszajn-Gagliardo for the real methods in the case of n-tuple of quasi-Banach.

1. Introduction and preliminaries

Let $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ be an n-tuple of quasi-Banach spaces and

$$\bar{t} = (t_1, t_2, \cdots, t_n) \in R_+^{n-1}.$$

The Peetre' K-functional is defined for $a \in \sum(\bar{A}) := A_0 + A_1 + \cdots + A_n$ by
$K(t_1, t_2, \cdots, t_n, a; \bar{A})$

$$= \inf\{\|a_0\|_{A_0} + t_1\|a_1\|_{A_1} + \cdots + t_{n-1}\|a_{n-1}\|_{A_{n-1}} \ : \ a = \sum_{i=0}^{n-1} a_i, \ a_j \in A_j\}.$$

Similarly the J-functional for $a \in \triangle(\bar{A}) := A_0 \cap A_1 \cdots \cap A_{n-1}$ by

$$J(t_1, t_2, \cdots, t_{n-1}, a; \bar{A}) = max\{\|a\|_{A_0}, t_1\|a\|_{A_1}, \cdots, t_{n-1}\|a\|_{A_{n-1}} \ : \ a \in \triangle(\bar{A})\}.$$

Let $\bar{A} = (A_0, A_1, A_2, \cdots, A_{n-1})$ be an n-tuple of quasi-Banach spaces and

$$\bar{n} = (n_1, n_2, \cdots, n_{n-1}) \in Z^{n-1}.$$

For $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} < 1, \theta_1 + \theta_2 + \cdots + \theta_{n-1} < 1$ and $0 < q \leq \infty$ we define the real interpolation space $\bar{A}_{(\theta_1, \theta_2, \cdots, \theta_{n-1}), q, K}$ as the set of all $a \in \sum(\bar{A})$ having a finite

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quasi-norm
$\|a\|_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,K} =$

$$\begin{cases} \left(\sum_{\bar{n}\in Z^{n-1}} (2^{\sum_{i=1}^{n-1} -n_i\theta_i} K(2^{n_1}, 2^{n_2}, \cdots, 2^{n_{n-1}}, a; \bar{A}))^q \right)^{1/q} & if\ 0 < q < \infty \\ \sup_{\bar{n}\in Z^{n-1}} \{2^{\sum_{i=1}^{n-1} -n_i\theta_i} K(2^{n_1}, 2^{n_2}, \cdots, 2^{n_{n-1}}, a; \bar{A})\} & if\ q = \infty \end{cases} .$$

Also we define the real interpolation space $\bar{A}_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,J}$ as the set of all $a \in \sum(\bar{A})$ which can be written as $a = \sum_{\bar{n}\in Z^{n-1}} u_{\bar{n}},\ u_{\bar{n}} \in \triangle(\bar{A})$ (convergence in $\sum(\bar{A})$) with a finite quasi-norm
$\|a\|_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,J}$

$$= \inf_{a=\sum_{\bar{n}\in Z^{n-1}} u_{\bar{n}}} \left(\sum_{\bar{n}\in Z^{n-1}} (2^{\sum_{i=1}^{n-1} -n_i\theta_i} J(2^{n_1}, 2^{n_2}, \cdots, 2^{n_{n-1}}, u_{\bar{n}}; \bar{A}))^q \right)^{1/q}$$

With the usual interpretation when $q = \infty$.

If $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ and $\bar{B} = (B_0, B_1, \cdots, B_{n-1})$ are n-tuple Banach spaces, we write $T \in \pounds(\bar{A}, \bar{B})$ to mean that T is a linear operator from $\sum(\bar{A})$ into $\sum(\bar{B})$ whose restriction to each A_j defines a bounded operator from A_j intoB_j $(j = 0, 1, \cdots, n-1)$. We put

$$\|T\|_{\bar{A},\bar{B}} = \max_{j=0,1,\cdots,n-1} \{\|T\|_{A_j,B_j}\}.$$

Scalar sequence spaces are defined over Z^{n-1} and given any sequence of positive numbers $(w_{\bar{n}})_{\bar{n}\in Z^{n-1}}$. Put

$$l_p(w_{\bar{n}}) = \{(a_{\bar{n}}) : \|a_{\bar{n}}\|_{l_p(w_{\bar{n}})} = \|w_{\bar{n}} a_{\bar{n}}\|_{l_p} < \infty\}.$$

Of special interest for us are the n-tuple $\bar{l}_p = \Big(l_p, l_p(2^{-n_1}), \cdots, l_p(2^{-n_{n-1}})\Big)$, $(0 < p \leq 1)$ and $\bar{l}_\infty = \Big(l_\infty, l_\infty(2^{-n_1}), \cdots, l_\infty(2^{-n_{n-1}})\Big)$.

A maximal description in sense of Aronszajn-Gagliardo [1] for the real method in the case of quasi-Banach couples is given in [2].

2. Minimal description

Let $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ be an n-tuple of quasi-Banach spaces. Recall that a quasi-norm $\|.\|$ is said to be a $p-$norm $(0 < p \leq 1)$ if

$$\|a + b\|^p \leq \|a\|^p + \|b\|^p$$

Here p is defined by the equation $(2c)^p = 2$, where c is the constant in the triangle inequality $\|.\|$.
Note also that if $\|.\|$ is a p-norm then it is also an r-norm for any $0 < r \leq p$.

Definition 2.1. Let $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} < 1, \theta_1 + \theta_2 + \cdots + \theta_{n-1} < 1$ and $0 < q \leq \infty$. Assume that $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ be an n-tuple of p-Banach spaces $(0 < p \leq 1)$. Put $r = min(p, q)$ and define $G_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,r}(\bar{A})$ as the collection of all $a \in \sum(\bar{A})$

which can be represented as a convergent series $a = \sum_{j=1}^{\infty} T_j a_j$ in $\sum(\bar{A})$ with $a_j \in l_q(2^{\sum_{i=1}^{n-1} -n_i\theta_i}), T_j \in \pounds(\bar{l}_p, \bar{A})$ and

$$\left(\sum_{j=1}^{\infty} \|T_j\|^r_{l_{\bar{p}},\bar{A}} \|a\|^r_{l_q(2^{\sum_{i=1}^{n-1} -n_i\theta_i})}\right)^{1/r} < \infty.$$

This spaces become an r-Banach space endowed with the functional

$$\|a\|_{G_{(\theta_1,\theta_2,\cdots,\theta_{n-1})q,r}}(\bar{A}) = inf\{\left(\sum_{j=1}^{\infty} \|T_j\|^r_{l_{\bar{p}},\bar{A}} \|a\|^r_{l_p}(2^{\sum_{i=1}^{n-1} -n_i\theta_i})\right) : a = \sum_{j=1}^{\infty} T_j a_j\}$$

Theorem 2.2. *Let $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ be an n-tuple of p-Banach spaces, let $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} < 1, \theta_1 + \theta_2 + \cdots + \theta_{n-1} < 1$ and $0 < q \leq \infty$. Put $r = min(p, q)$. Then*

$$(A_0, A_1, \cdots, A_{n-1})_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,J} = G_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,r}(A_0, A_1, \cdots, A_{n-1})$$

3. Maximal description

Let T be a mapping from a quasi-Banach space A into a scalar sequence space M. We say that T is quasi$-$linear with constant $C \geq 1$ if

$$|T(a+b)| \leq C\left(|Ta| + |Tb|\right), \qquad a, b \in A$$

$$|T(\lambda a)| = |\lambda||Ta|, \quad a \in A, \quad \lambda \in F,$$

where F is the scalar field. Given any n-tuple of quasi-Banach spaces $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ and $C \geq 1$ we denote by $\pounds_C(\bar{A}, \bar{l}_\infty)$ the collection of all those quasi-linear operators $T : \sum(\bar{A}) \to \sum(\bar{l}_\infty)$ with the constant C whose restriction to A_i $(i = 0, 1, \cdots, n-1)$ defines a bounded operator from $A_0, A_1, \cdots, A_{n-1}$ into $l_\infty, l_\infty(2^{-n_1}), l_\infty(2^{-n_2})$ $, \cdots, l_\infty(2^{-n_{n-1}})$, respectively.

Definition 3.1. Let $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} < 1, \theta_1 + \theta_2 + \cdots + \theta_{n-1} < 1$ and $0 < q \leq \infty$. Given any n-tuple of quasi-Banach spaces $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ we define $H_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,C}(\bar{A})$ as the collection of all $a \in \sum(\bar{A})$ such that $Ta \in l_q(2^{\sum_{i=1}^{n-1} -n_i\theta_i})$ for any $T \in \pounds_C(\bar{A}, \bar{l}_\infty)$ and quasi-norm

$$\|a\|_{H_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,C}}(\bar{A}) = sup\{\|Ta\|_{l_q(2^{\sum_{i=1}^{n-1} -n_i\theta_i})} : \|T\|_{\bar{A},\bar{l}_\infty} \leq 1\}$$

is finite.

Theorem 3.2. *Let $\bar{A} = (A_0, A_1, \cdots, A_{n-1})$ be a quasi-Banach n-tuple, let $0 < \theta_1, \theta_2, \cdots, \theta_{n-1} < 1, \theta_1 + \theta_2 + \cdots + \theta_{n-1} < 1$ and $0 < q \leq \infty$. Assume that the constant in the triangle inequality of A_i is C_i $(i = 0, 1, \cdots, n-1)$ and put $C = max(C_0, C_1, \cdots, C_{n-1})$. Then*

$$(A_0, A_1, \cdots, A_{n-1})_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,K} = H_{(\theta_1,\theta_2,\cdots,\theta_{n-1}),q,C}(A_0, A_1, \cdots, A_{n-1}).$$

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ANALYTIC PROJECTIONS ON SPECIAL SEQUENCE SPACES

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ABSTRACT. We give sufficient conditions for the boundedness of the analytic projection on the set of multipliers of the Banach space of formal Laurent series. This presents the sufficient conditions to a problem that has considered by A. L. Shields.

1. INTRODUCTION AND PRELIMINARIES

Let $\{\beta(n)\}_{n=-\infty}^{\infty}$ be a sequence of positive numbers satisfying $\beta(0) = 1$. If $1 \le p < \infty$, the space $L^p(\beta)$ consists of all formal Laurent series $f(z) = \sum_{n=-\infty}^{\infty} \hat{f}(n)z^n$ such that the norm $\|f\|^p = \|f\|_\beta^p = \sum_{n=-\infty}^{\infty} |\hat{f}(n)|^p \beta(n)^p$ is finite. When n just runs over $\mathbb{N} \cup \{0\}$, the space $L^p(\beta)$ only contains formal power series $f(z) = \sum_{n=0}^{\infty} \hat{f}(n)z^n$ and it is usually denoted by $H^p(\beta)$. We denote the set of multipliers $\{\varphi \in L^p(\beta) : \varphi L^p(\beta) \subseteq L^p(\beta)\}$ by $L^p_\infty(\beta)$. Let $\varphi(z) = \sum_{k=-\infty}^{\infty} \hat{\varphi}(k)z^k = \varphi_1(z) + \varphi_2(z)$ be in $L^p_\infty(\beta)$, where

$$\varphi_1(z) = \sum_{k=0}^{\infty} \hat{\varphi}(k)z^k, \qquad \varphi_2(z) = \sum_{k=1}^{\infty} \hat{\varphi}(-k)z^{-k}.$$

Define the analytic projection $J : L^p_\infty(\beta) \longrightarrow L^p(\beta)$ by $J(\varphi) = \varphi_1$. This projection is not necessarily a bounded operator on $L^p_\infty(\beta)$. For example J is not bounded when $\beta(n) = 1$ for all n in $\mathbb{Z}$. We want to investigate that for which multiplication operators the analytic projection J is a bounded linear operator from $L^p_\infty(\beta)$ into $L^p_\infty(\beta)$ and we present a sufficient conditions to the following question that has considered by Shields:

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* Speaker.

Question. For which bilateral shifts the analytic projection is a bounded operator on $L^2_\infty(\beta)$?

2. Main results

We will use the notations of [6]. For other main references see [1-5].

Lemma 2.1. *If there exists a constant $c > 0$ such that $\|M_{J(p)}\| \leq c\|M_p\|$ for all Laurent polynomials p, then $J \in B(L^p_\infty(\beta))$.*

Theorem 2.2. *Let M_z be invertible on $L^p(\beta)$, $r_{22} < r_{33}$ and let for some $d > 0$, $\|p\|_{\Omega_3} \leq d\|M_p\|$ for all Laurent polynomials p. Then $J \in B(L^p_\infty(\beta))$.*

Theorem 2.3. *Let M_z be invertible on $L^p(\beta)$, $r_{23} < r_{12}$ and let for some $d > 0$, the relation $\|q\|_{\Omega_3} \leq d\|M_q\|$ holds for all Laurent polynomials q. Then $J \in B(L^p_\infty(\beta))$.*

Theorem 2.4. *Let M_z be invertible on $L^p(\beta)$, $r_{12} < r_{22}$ and $\sigma(M_z)$ be a spectral set for M_z. Then $J \in B(L^p_\infty(\beta))$.*

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REPRESENTATION OF SOLUTION FOR THE NONLINEAR OPERATOR EQUATIONS IN THE REPRODUCING KERNEL HILBERT SPACES

MAHMOUD HADIZADEH YAZDI*

Abstract. A reproducing kernel Hilbert space restricts the space of functions to smooth functions and has the ideal structure for function approximation and some aspects in learning theory. We discuss the structure of the solution space of some nonlinear operator equations in reproducing kernel Hilbert space. Actually, if the solution exists, we give the analytic representation of the minimal normal solution of the problem. We also try to review some of the basic facts regarding the reproducing kernel Hilbert spaces and their applications in approximation theory, learning theory as well as representation of the exact solution of some classes of nonlinear integral equations.

1. Introduction and preliminaries

Exact solutions have always played an important role in properly understanding the qualitative features of many phenomena and processes in various fields of natural science. It is natural to adopt, as the criterion of simplicity, the requirement that the model equation admits a solution in a closed form. Moreover, the model equations and problems admitting exact solutions serve as the basis for the development of new numerical, asymptotic and approximate methods which in turn, enable us to investigate more complicated problems having no analytical solution.

Reproducing Kernel Hilbert Spaces (RKHS) are wonderful objects and can be used in a wide variety of curve fitting, function estimation and model description. It is well known that linear algebra provides us with powerful tools for thinking about learning algorithms. We will focus on Hilbert spaces which allow defining the fundamental

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* Invited Speaker.

notations of length, distance, orthogonality, projection and similarity in a very general domain-independent manner. We show that using the concept of a generalized inner product, we can design algorithms for arbitrary data sets, including strings, graphs and rational domains. Nowadays, kernel methods is one of the fastest growing and most exciting areas in machine learning, but it needs a deeper knowledge of some aspects in mathematical analysis. Note that, in many applications it is useful to map the input observations x_j first into an element $\Phi(x_j)$ of a larger "feature space" describing additional properties of x_j, in order to be able to use a specific distance $dist(\Phi(x_j), \Phi(x_k))$ in feature space that allows to model the similarity of x_j with x_k much more closely than by the data x_j and x_k themselves. This key idea was termed "ernel trick" by the community focusing on "learning algorithm" in 2002 by Schölkopf and Smola [9], but its mathematical roots date back to reproducing kernel Hilbert spaces in 1950 by Aronszajn [1]. The feature map Φ is supposed to map into some function space $\mathbb{F}$ that carries an inner product $<,>$ and a reproducing kernel K such that :

$$\Phi(x) = K(x,.)$$

$$< \Phi(x), \Phi(y) >=< K(x,.), K(y,.) >= K(x,y)$$

$$< \Phi(x), v >=< K(x,.), v >= v(x),$$

for all $x, y \in \mathbb{R}^n$ and all $v \in \mathbb{F}$. Such kernels exists in many variations and the space $\mathbb{F}$ can be written as the Hilbert space closure of all images $\Phi(x) = K(x,.)$ of the feature map under the above inner product.

Definition 1.1. Given a set X, we will say that H is a reproducing kernel Hilbert space (RKHS) on X over $\mathbb{F}$, provided that:
(i) H is a vector subspace of the set of all functions from X to $\mathbb{F}$.
(ii) H is endowed with an inner product $<,>$, making it into a Hilbert space.
(iii) for every $y \in X$, the linear evaluation functional $E_y : H \to \mathbb{F}$, defined by $E_y(f) = f(y)$, is bounded.

Let H be an RKHS on X. Since every bounded linear functional is given by the inner product with a unique vector in H, for every $y \in X$, there exists a unique vector $k_y \in H$, such that for every $f \in H$, $\quad f(y) =< f, k_y >$.

Definition 1.2. The function k_y is called the reproducing kernel for the point y and the two variable function defined by

$$K(x, y) = k_y(x),$$

is called the reproducing kernel for H.

Note that we have:

$$K(x, y) = k_y(x) =< k_y, k_x >,$$

and

$$\|k_y\|^2 =< k_y, k_y >= K(y, y).$$

Theorem 1.3. *(The Moore-Aronszajn theorem) [1]: For every positive definite function $K(.,.)$ on $X \otimes X$, there exists a unique RKHS and vise versa.*

The Hilbert space associated with K can be constructed as the space of all finite linear combination of the form $\sum a_j K(x_j, .)$, and their limits under the norm induced by the inner product $< K(x, .), K(y, .) >= K(x, y)$. Norm convergence implies pointwise convergence in a RKHS, as can be seen by observing that

$$|f_n(y) - f_m(y)| = | < K(y, .), f_n - f_m > | \leq K(y, y)\|f_n - f_m\|.$$

So, these limit functions are well defined pointwise.

2. Representation of the solution

In this section, we give a procedure which transforms any nonlinear operator equation into a linear equation and so we discuss the problem of how solving the continuous linear operator equations in a separable Hilbert space.

Let H be a separable Hilbert space, A be a bounded linear operator from H into H and consider the linear operator equation: $Au = f, \quad u, f \in H$.

During the last 5 years, significant progress has been made in applications of the RKHS in some classes of the mentioned operator equation, e.g. in singular boundary value problems [7], system of second order boundary value problems [4], partial differential equations [8], system of ill-posed operator equations of the first kind [2] and some relevant works in [5,3,6,10]. The representation of the exact solution of the nonlinear integral equations will be obtained in the reproducing kernel space. The exact solution is given by the form $u = \sum c_n s_n$, where the coefficients c_n and harmonics s_n have computable and relevant properties, optimal and desired duration properties in terms of the information area, respectively.

Consider the following nonlinear Volterra-Feredholm integral equations of the form:

$$u(x, t) = f(x, t) + \lambda \int_0^t \int_{\Omega'} F(x, t, \xi, \tau, u(\xi, \tau)) d\xi d\tau, \quad (x, t) \in [0, T] \times \Omega',$$

where $u(x, t)$ is an unknown function, $f(x, t)$ and $F(x, t, \xi, \tau, u(\xi, \tau))$ are given analytical real-valued functions defined respectively on $D = [0, T] \times \mathbb{R}$ and $S \times \mathbb{R}$, where $S = \{(x, t, \xi, \tau), \quad 0 \leq \tau \leq T, \quad (x, t) \in \Omega' \times \Omega'\}$, and λ is a real constant, with $F(x, t, \xi, \tau, u(\xi, \tau))$ nonlinear in u and Ω' is a closed subset of $\mathbb{R}^n$.

For simplicity, let $\Omega' = [a, b]$ and $\Omega = [0, T] \times [a, b]$ in $\mathbb{R}^2$ and suppose

$$W_2^2(\Omega) = \left\{ u(x, t) |\ u(x, t) \text{ is absolutly continuous and} \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}, \frac{\partial^2 u}{\partial x \partial t} \in L^2(\Omega) \right\}.$$

We now define, for every $u, v \in W_2^2(\Omega)$,

$$< u, v >= \int \int_\Omega \left(u(x, t)v(x, t) + \frac{\partial u}{\partial x}\frac{\partial v}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial v}{\partial t} + \frac{\partial^2 u}{\partial x \partial t}\frac{\partial^2 v}{\partial x \partial t} \right) d\sigma$$

as inner product on $W_2^2(\Omega)$ and norm $\|u\|_{W_2^2} =< u, u >^{\frac{1}{2}}$.

Now, let the integral operator N be defined as

$$(Nu)(x, t) = \int_0^t \int_a^b F(x, t, \xi, \tau, u(\xi, \tau)) d\xi d\tau,$$

so that the main equation becomes

$$u(x, t) = f(x, t) + \lambda (Nu)(x, t).$$

Let $\psi_i(x,t) = k_{(\xi_i,\tau_i)}(x,t)$, such that

$$k_{(\xi,\tau)}(x,t) = k_\xi(x)k_\tau(t),$$

where $k_{(\xi,\tau)}(.,.)$ and $k_\xi(.)$ are the reproducing kernels of the spaces $W_2^2(\Omega)$ and $W_2^1(\Omega')$, respectively.

Now, we orthonormalize the functions system $\{\varphi_i(x,t)\}_{i=1}^\infty$, and so

$$\overline{\psi}_i(x,t) = \sum_{k=1}^{i} \beta_{ik}\psi_k(x,t), \qquad i = 1, 2, ...$$

Lemma 2.1. *If $\{p_i\}_{i=1}^\infty$ is a given dense sequence in Ω, then $\{\psi_i(x,t)\}_{i=1}^\infty$ is the complete function system of the space $W_2^2(\Omega)$.*

Using the mentioned Lemma, we have the following characterization of reproducing kernels, which is our main result:

Theorem 2.2. *Let $\{p_i\}_{i=1}^\infty$ is dense in Ω. If the main equation has a unique solution, then the exact solution u is decomposed by reproducing kernels:*

$$u(x,t) = \sum_{i=1}^{\infty}\sum_{k=1}^{i} \beta_{ik}(f(x_k,t_k) + \lambda(Nu)(x_k,t_k))\overline{\psi}_i(x,t).$$

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MACHINERY OF ROSSER-SCHOENFELD METHOD FOR EXPLICIT APPROXIMATING OF PRIMES

MEHDI HASSANI

Abstract. We introduce the machinery of a method due to Rosser and Schoenfeld for approximating Chebychev functions in primenumberology, which ends to explicit approximation of primes. The story backs to the Riemann's work on prime numbers. He guessed an explicit formula between the Chebychev function $\psi(x) = \sum_{p^m \leq x} \log p$ and nontrivial zeros $\rho = \beta + i\gamma$ of the Riemann zeta function, defined for $Re(s) > 1$ by $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ and extended by meromorphic continuation to the complex plan. The connection (which is known as Riemann's explicit formula) includes some elementary functions and the strange term $\lim\limits_{T\to\infty} \sum_{|\gamma|\leq T} \frac{x^\rho}{\rho}$, and aim of Rosser-Schoenfeld method is approximating this summation. The procedure of method, needs some numerical and approximate data about non-trivial zeros of Riemann zeta function, which force us studying and applying zero free regions of this function. Also, it requires using complex analytic methods and theory of special functions.

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HYERS-ULAM STABILITY ON MEASURABLE FUNCTION SPACES

MOHAMMAD REZA JABBARZADEH

Abstract. For a weighted composition operator $uC_\varphi : f \longmapsto u.(f \circ \varphi)$ on $L^p(\Sigma)$ spaces, we give a necessary and sufficient condition for uC_φ to have the Hyers-Ulam stability, in terms of conditional expectation operator.

1. Preliminaries And Notations

Let B be a Banach space and let T be a mapping from B into itself. We say that T has the Hyers-Ulam stability, if there exists a constant K with the following property:

(a) For any $g \in T(B)$, $\varepsilon > 0$ and $f \in B$ satisfying $\|Tf - g\| \leq \varepsilon$, we can fined an $f_0 \in B$ such that $Tf_0 = g$ and $\|f - f_0\| \leq K\varepsilon$.

We call K a HUS constant for T, and denote the infimum of all HUS constants for T by K_T. A subspace M of B is said to be proximinal, if for any $f \in B$, there exists $g \in M$ such that $\|f - g\| = \|f + M\|$. Some good sources about the Hyers-Ulam stability of substitution operators on function spaces are [1], [2], [3], [4], [5] and [6].

From now on, by an operator we shall mean a non-zero linear operator. The linearity of T implies that the condition (a) is equivalent to the fact that for any $\varepsilon > 0$ and $f \in B$ with $\|Tf\| \leq \varepsilon$ there exists an $f_0 \in B$ such that $Tf_0 = 0$ and $\|f - f_0\| \leq K\varepsilon$. For a bounded operator $T : B \to B$, we denote the null space of T by $N(T)$ and the range of T by $R(T)$. When T is not one-to-one, one may consider the operator $\widetilde{T}$ from $B/N(T)$ into B defined by $\widetilde{T}(f + N(T)) = Tf$, for all $f \in B$. Clearly $\widetilde{T}$ is a one-to-one operator and $R(\widetilde{T}) = R(T)$.

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Let (X, Σ, μ) be a complete σ-finite measure space. Let φ be a measurable transformation from X into X and $\mu(\varphi^{-1}) \ll \mu$. To examine the weighted composition operators efficiently, Lambert in [**?**] associated with each transformation φ, the so-called conditional expectation operator $E(\bullet|\varphi^{-1}(\Sigma)) = E(\bullet)$. $E(f)$ which is defined for each non-negative measurable function f or for each $f \in L^p(\Sigma)$, and is uniquely determined by the conditions:

(i) $E(f)$ is $\varphi^{-1}(\Sigma)$-measurable and
(ii) If A is any $\varphi^{-1}(\Sigma)$-measurable set for which $\int_A f d\mu$ converges, we have

$$\int_{\varphi^{-1}(A)} f d\mu = \int_{\varphi^{-1}(A)} E(f) d\mu.$$

If $u : X \to \mathbb{C}$ is a measurable function, the weighted composition operator uC_φ on $L^p(\Sigma)$ induced by φ and u is given by

$$uC_\varphi(f) = u.f \circ \varphi, \qquad f \in L^p(\Sigma).$$

2. Main results

In this paper for a weighted composition operator $uC_\varphi : L^p(\Sigma) \to L^p(\Sigma)$, we give a necessary and sufficient condition for uC_φ to have the Hyers-Ulam stability and then we show that K_{uC_φ} is a HUS constant for uC_φ.

Theorem A. Let $1 \le p < \infty$ and Σ be φ-invariant. If uC_φ is a bounded weighted composition operator on $L^p(\Sigma)$, then the following assertions are equivalent:
(i) uC_φ has the Hyers-Ulam stability;
(ii) uC_φ has closed range;
(iii) there exists $r > 0$ such that $J(x) := (h(x)E(|u|^p) \circ \varphi^{-1}(x))^{1/p} \ge r$ for μ-almost all $x \in \sigma(J)$;
(iv) there exists $r > 0$ such that $\varphi(\sigma(u)) \subseteq \{x \in X : J(x) \ge r\}$;
(v) there exists $K > 0$ such that $\|f + N(uC_\varphi)\| \le K\|uC_\varphi f\|$, for all $f \in L^p(\Sigma)$.

The following theorem shows that the HUS constant for uC_φ is $1/R$, where $R = \sup\{r > 0 : \varphi(\sigma(u)) \subseteq \{J \ge r\}\}$.

Theorem B. If $R = \sup\{r > 0 : \varphi(\sigma(u)) \subseteq \{J \ge r\}\}$, then $K_{uC_\varphi} = 1/R$.

Corollary C. Suppose $u \ne 0$ and C_φ is a one-to-one operator on $L^p(\Sigma)$. If uC_φ is a bounded operator on $L^p(\Sigma)$, then uC_φ has the Hyers-Ulam stability if and only if uC_φ is a Fredholm operator.

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WEIGHTED COMPOSITION OPERATORS ON WEIGHTED HARDY SPACES

SEDIGHEH JAHEDI

Abstract. This paper gives sufficient conditions to determine when the adjoint of a weighted composition operator on a weighted Hardy space is a composition operator. Also we will give sufficient conditions for hyponormality of the adjoint of a weighted composition operator.

1. Introduction

Let u be an analytic function on the open unit disk and φ be an analytic selfmap of the unit disk. A weighted composition operator $C_{u,\varphi}$ maps an analytic function f on the unit disk D into

$$(C_{u,\varphi}f)(z) = u(z)f(\varphi(z)).$$

φ is called the composition map and u is a weight. If $u \equiv 1$ then $C_{u,\varphi}$ is a composition operator. To avoid $C_{u,\varphi}$ being a multiplication operator, the composition map φ is taken to be different from identity. Let $\{\beta(n)\}$ be a sequence of positive numbers, and $1 \leq p < \infty$. Let $f = \{\hat{f}(n)\}$ be such that

$$\|f\|^p = \|f\|_\beta^p = \sum_{n=0}^{\infty} |\hat{f}(n)|^p \beta(n)^p < \infty.$$

The notation $f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n$ shall be used wether or not the series converges for any value of z. The set $H^p(\beta)$, $1 \leq p < \infty$ of these formal power series with $\|f\|_\beta^p < \infty$ is called weighted Hardy space. These notations are suggested by the coming in [2]. In the case $p = 2$, the classical Hardy space, the classical Bergman space, the classical Dirichlet space are weighted Hardy spaces with $\beta(n) \equiv 1$, $\beta(n) = (n+1)^{-\frac{1}{2}}$ and $\beta(n) = (n+1)^{\frac{1}{2}}$, respectively. These spaces are reflexive Banach spaces with the

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$\|.\|_\beta$. [1, 3]. Suppose that $\liminf_{n\longrightarrow\infty} \beta(n)^{\frac{1}{n}} \geq 1$. A complex number is said to be a bounded point evaluation on $H^p(\beta)$ if the functional of point evaluation at λ , e_λ, is bounded.

2. Main Results

Lemma 2.1. *If $C_{u,\varphi}$ is a bounded weighted composition operator on a weighted Hardy space $H^p(\beta)$, $1 \leq p < \infty$,$\frac{1}{p} + \frac{1}{q} = 1$, and e_λ is bounded at λ then*

$$C^*_{u,\varphi} e_\lambda = \overline{u(\lambda)} e_{\varphi(\lambda)}.$$

Theorem 2.2. *Let $A = C_{u,\varphi}$ be a weighted composition operator on $H^2(\beta)$. If A^* is a composition operator then $u = \alpha e_\lambda$ for some α and λ in the open unit disk and $\varphi(0) = 0$.*

Proposition 2.3. *If $C_{u,\varphi}$ is bounded on $H^p(\beta)$, $1 \leq p < \infty$, then $|u(z)| \leq \|C_{u,\varphi}\| \frac{\|e_z\|}{\|e_{\varphi(z)}\|}$ for z in the open unit disk.*

Theorem 2.4. *Suppose that $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q} < \infty$ and the weighted composition operator $C_{u,\varphi}$ is bounded on $H^p(\beta)$. Then $\|C_{u,\varphi}\|$ is bounded below by $\frac{\|u\|_\infty}{\beta(0)(\sum_{n=0}^{\infty} \frac{1}{\beta(n)^q})^{\frac{1}{q}}}$.*

Proposition 2.5. *Let u be an analytic selfmap on the open unit disk with $u(0) \neq 0$. Let $\{\beta_n\}$ be such that $\beta(0) = 1$ and $\sum_{n=0}^{\infty} \frac{1}{\beta(n)^2} < \infty$. If $C^*_{u,\varphi}$ is hyponormal on $H^2(\beta)$ then $\varphi(0) = 0$.*

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PERTURBATIONS OF BANACH ALGEBRAS AND ALMOST MULTIPLICATIVE FUNCTIONALS

KRZYSZTOF JAROSZ*

Abstract. We discuss small deformations of Banach algebras and multiplicative functionals. We are particularly interested in properties that remain stable under such small deformations (perturbations). We survey the history of that area, provide some recent results, and discuss numerous open problems.

1. Introduction

Let G be a linear and multiplicative functional on a Banach algebra A and let Δ be a linear functional on A with $\|\Delta\| \leq \varepsilon$. Put $F = G + \Delta$. We can easily check by direct computation that F is *δ-multiplicative,* that is,

$$|F(ab) - F(a)F(b)| \leq \delta \|a\| \|b\|, \qquad \text{for } a, b \in A,$$

where $\delta = 3\varepsilon + \varepsilon^2$. The problem we want to discuss here is whether the converse is true; that is, whether an almost multiplicative functional must be near a multiplicative one. We are interested mostly in uniform algebras. We shall call a Banach algebra *functionally-stable* or *f-stable* if

$$\forall \varepsilon > 0 \ \exists \delta > 0 \ \forall F \in \mathfrak{M}_\delta(A) \ \exists G \in \mathfrak{M}(A) \ \ \|F - G\| \leq \varepsilon,$$

where we denote by $\mathfrak{M}(A)$ the set of all linear multiplicative functionals on A, and by $\mathfrak{M}_\delta(A)$ the set of δ-multiplicative functionals on A.

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* Invited Speaker.

2. History

The question whether an almost multiplicative map is close to a multiplicative one constitutes an interesting problem per se; nevertheless, it originated in the deformation theory of Banach algebras. There are two basic concepts of deformation of Banach algebras: metric and algebraic [1].

Definition 2.1. We say that a Banach algebra B is a metric δ-deformation of a Banach algebra A if there is a linear (but not necessarily multiplicative) isomorphism $T : A \to B$ such that $\|T\| \left\|T^{-1}\right\| \leq 1 + \delta$.

Definition 2.2. For a Banach algebra $(A, \cdot)$ we say that a new multiplication $\times$ defined on the same Banach space A is an algebraic δ-deformation of $(A, \cdot)$ if $\|\times - \cdot\| \leq \delta$; that is, if

$$\|a \cdot b - a \times b\| \leq \delta \|a\| \|b\|, \qquad \text{for } a, b \in A.$$

While the two definitions lead to different theories for general Banach algebras, they are equivalent in a natural way for all uniform algebras [1]. In particular:

- **(i):** two uniform algebras are isometric if and only if they are isomorphic as algebras,
- **(ii):** a linear map $T : A \to B$ between uniform algebras almost preserves the distance if and only if it almost preserves the multiplication of the algebras.

There are several important links between the deformation theory and other areas. For example it provides a natural definition of deformation of an analytic manifold, or a domain Ω in $\mathbb{C}^n$. We may define the distance between two domains Ω and Ω' by

$$d(\Omega, \Omega') = \inf\left\{\|T\| \left\|T^{-1}\right\| : \quad T : A(\Omega) \to A(\Omega')\right\},$$

where $A(\Omega)$ is a Banach space of analytic functions on Ω. It is an important and deep result due to R. Rochberg [2] that for one dimensional Riemann surfaces the distance defined above is locally equivalent to the Teichmüller distance involving quasiconformal homeomorphisms. Still, almost nothing is known about domains in $\mathbb{C}^n$ for $n > 1$.

If $\times$ is a small algebraic deformation of a Banach algebra $(A, \cdot)$, then any multiplicative functional on A is almost $\times$-multiplicative. Since the main objective of the deformation theory of Banach algebras is to compare structures of two close algebras we would like to know if an almost multiplicative functional must be close to a multiplicative one.

It is not difficult to show that the class of $C(K)$ algebras is uniformly-f-stable. In 1986 B. E. Johnson also proved that the disc algebra $A(\mathbb{D})$ and some related algebras are f-stable (Johnson uses the name *AMNM* algebras). It was then conjectured that all uniform algebras are f-stable. However, very recently S. J. Sidney provided an ingenuous counterexample [3].

We will discuss *f-stability* for general uniform algebras, primarily for uniform algebras. we prove that any uniform algebra with one generator as well as some algebras of the form $R(K)$, $K \subset \mathbb{C}$ and $A(\Omega)$, $\Omega \subset \mathbb{C}^n$ are *f-stable*. We show that, for a Blaschke product B, the f-stability of the quotient algebra H^∞/BH^∞ is related to the distribution of zeros of B. For example, if B is not a product of finitely many interpolating Blaschke products then H^∞/BH^∞ is not *f-stable.*

It is still an open problem if $H^\infty(\mathbb{D})$ is f-stable. In view of the importance of the Corona Theorem for $H^\infty(\mathbb{D})$ it is particularly interesting to know whether $H^\infty(\mathbb{D})$ does not have an *almost corona,* that is, a set of almost multiplicative functionals far from the set of multiplicative functionals.

3. Open Problems

Problem 1.: Is H^∞ f-stable?

Problem 2.: Let B be a product of finitely many interpolating Blaschke products. Is H^∞/BH^∞ f-stable?

Problem 3.: Let K be a compact subset of the complex plane. Is $R(K)$ f-stable? Is $H^\infty(K)$ f-stable?

Problem 4.: Let Ω be a bounded pseudoconvex domain in C^n. Are $A(\Omega)$ and $H^\infty(\Omega)$ f-stable?

Problem 5.: Is any uniform algebra with two generators f-stable?

Problem 6.: Let A be an f-stable uniform algebra. Is the algebra

$$l^\infty(A) = \left\{ (f_n)_{n=1}^\infty : \forall n \ f_n \in A, \text{ and } \|(f_n)\| = \sup_n \|f_n\| < \infty \right\}$$

f-stable?

Problem 7.: Let A be an f-stable uniform algebra. Is an ultrapower of A f-stable?

Problem 8.: Let A be a uniform algebra such that the family of all quotient algebras A/I, is uniformly f-stable, where I is a closed ideal in A. Is $A = C(K)$ for some compact set K?

Problem 9.: Characterize bounded pseudoconvex domains Ω be in C^n such that $A(\Omega)$ and $H^\infty(\Omega)$ stable.

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WEIGHTED SEMITOPOLOGICAL SEMIGROUPS AND THEIR COMPACTIFICATIONS

ALI AKBAR KHADEM MABOUDI

Abstract. In the recent years, a number of authors have studied various types of functions and measures on weighted semigroups (specially weighted groups). The general theorems on which this talk is based are the study of relations of compactifications of a weighted semitopological semigroup (S, w), and m-admissible subalgebras of $C(S, w)$, the C^*-algebra of all continuous bounded functions on S with respect to $\|f\|_w := \|f/w\|_\infty$. The present correspondence plays a pivotal role for the study of the involved function algebras.

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A CLASS OF BANACH SEQUENCE SPACES ANALOGOUS TO THE SPACE OF POPOV

ALIREZA AHMADI LEDARI

Abstract. Hagler and Azimi introduced a class of hereditarily l_1 Banach spaces which fail the Schur property. Then the first author extended these spaces to a class of hereditarily l_p Banach spaces for $1 \leq p < \infty$. Here we use these spaces to introduce a new class of hereditarily $l_p(c_0)$ Banach spaces analogous to the space of Popov. In particular, for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces failing the Schur property.

1. Introduction

A class of hereditarily l_1 Banach spaces has been introduced by Hagler and Azimi, which among the other interesting properties fails the Schur property [2]. Then these spaces have been extended to a new class of hereditarily l_p Banach spaces, the $X_{\alpha,p}$ [1]. In 2005, Popov constructed a new class of hereditarily l_1 subspace of L_1 without the Schur property [5] and generalized his result to a class of hereditarily l_p Banach spaces [6]. In this paper we use the $X_{\alpha,p}$ spaces [1] to introduce and study a new class of hereditarily l_p spaces, analogous to the space of Popov. In particular, we show that for $p = 1$ the spaces are further examples of hereditarily l_1 Banach spaces which fail the Schur property. This would be the fourth example of this type. The first was constructed by J. Bourgain [3], the second by Hagler and Azimi, and the third by Popov.

Our construction shows that for the case $p = 0$ the spaces are hereditarily c_0.

Before we define this new spaces let to recall the definition of the $X_{\alpha,p}$. Let (α_i) be a sequence of reals in $[0, 1]$ (whose terms are used as weighting factor in the definition of the norm) which satisfies the following properties:

(1) $1 = \alpha_1 \geq \alpha_2 \geq ...,$

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Key words and phrases. l_1 hereditarily, Banach sequence space, Schur property.

(2) $\lim_i \alpha_i = 0$,
and
(3) $\sum_{i=1}^{\infty} \alpha_i = \infty$.

By a block F we mean an interval (finite or infinite) of integers. For a block F and $x = (t_1, t_2, ...)$ a sequence of scalars such that $\sum_j t_j$ converges, define $< x, F >= \sum_{j \in F} t_j$. A sequence $F_1, F_2, ..., F_n, ...$where each F_i is a finite block is admissible if

$$\max F_i < \min F_{i+1} \text{ for } i = 1, 2, 3, ...$$

For $x = (t_1, t_2, ...)$ a finitely nonzero sequence of scalars, define

$$\| x \| = \max \left(\sum_{i=1}^{n} \alpha_i \, |< x, F_i >|^p\right)^{\frac{1}{p}},$$

where the max is taken over all n, and admissible sequences $F_1, F_2, ..., F_n$ and $1 \leq p < \infty$. Then $X_{\alpha,p}$ is the completion of the finitely nonzero sequences of scalars $x = (t_1, t_2, ...)$ in this norm. For a good information concerning these spaces, referred to [1] and [2].

Now we go through the construction of the spaces X_p analogous of the space of Popov. Let α be a fixed sequence, and $(X_{\alpha,p_n})_{n=1}^{\infty}$ a sequence of Banach spaces as above with $\infty > p_1 > p_2 > ... > 1$. The direct sum of these spaces in the sense of l_p is defined as the linear space $X_p = \left(\sum_{i=1}^{\infty} \oplus X_{\alpha,p_n}\right)_p$ with $p \in [1, \infty)$ which is the space of all sequences $x = \left(x^1, x^2, ...\right)$, $x^n \in X_{\alpha,p_n}, n = 1, 2, ...$ with $\| x \|_p = (\sum_{n=1}^{\infty} \| x^n \|_{\alpha,p_n}^p)^{\frac{1}{p}} < \infty$.

The direct sum of the spaces (X_{α,p_n}) in the sense of c_0 is the linear space

$$X_0 = \left(\sum_{n=1}^{\infty} \oplus X_{\alpha,p_n}\right)_0$$

of all sequences $x = \left(x^1, x^2, ...\right)$, $x^n \in X_{\alpha,p_n}$, $n = 1, 2, ...$for which $\lim_n \| x^n \|_{\alpha,p_n} = 0$ with the norm

$$\| x \|_0 = \max_n \| x^n \|_{\alpha,p_n}.$$

A Banach space X is hereditarily l_p if every infinite dimensional subspace of X contains a subspace isomorphic to l_p. A Banach space X has the Schur property if norm convergence and weak convergence coincide. It is well known that l_1 has the Schur property.

We follow the same notations and terminology as in [4]. The construction and idea of the proof follow [6] but the nature of these spaces are different, so for similar results we omit the detail of proofs. In fact these spaces are a rich class of spaces which depend on the sequences (α_i) and (p_n) as above.

Fix a sequence (α_i) of reals which satisfies the above conditions, and a sequence (p_n) of reals with $\infty > p_1 > p_2 > ... > 1$.Consider the sequence space X_p as above. For each $n \geq 1$, denote by $(\bar{e}_{i,n})_{i=1}^{\infty}$ the unit vector basis of X_{α,p_n} and by $(e_{i,n})_{i=1}^{\infty}$ its natural copy in X_p :

$$e_{i,n} = (\underbrace{0, ...0}_{n-1}, \bar{e}_{i,n}, 0, ...) \in X_p.$$

Let $\delta_n > 0$ and $\Delta = (\delta_n)$ such that $\sum_{i=1}^{\infty} \delta_n^p = 1$ if $p \geq 1$, and $lim_n \delta_n = 0$ and $\max_n \delta_n = 1$ if $p = 0$. For each $i \geq 1$ put $z_i = \sum_{n=1}^{\infty} \delta_n e_{i,n}$.

Let Z_p be the closed linear span of $(z_i)_{i=1}^{\infty}$. Here is the main result of this paper

Theorem 1.1.(i) *the Banach space Z_p is hereditarily l_p for $p > 1$*

(ii) for $p = 1$ the space Z_1 is hereditarily l_1 and fails the Schur property.

(iii) The space Z_0 is hereditarily c_0.

2. Some Results

For each $I \subseteq \mathbb{N}$ the projection P_I denotes the natural projection of X onto $[e_{i,n} : i \in \mathbb{N}, n \in I]$. For $n \in \mathbb{N}$ denote $Q_n = P_{\{n,n+1,...\}}$.

Lemma 2.1. *Let $p \geq 1$ and (v_i) be a sequence in $X_{\alpha,p}$, (G_i) an admissible sequence of blocks such that $\{j : v_i\,(j) \neq 0\} \subset G_i$ and*

1. $\| \, v_i \, \| \leq 2$,
2. $s\,(v_i) \to 0$.

Then

$$\| \sum_{i=1}^{k} t_i v_i \, \|^p \leq 2\,(3)^{p-1} \sum_{i=1}^{k} \mid t_i \mid^p .$$

Lemma 2.2. *Let (u_i) be a norm one sequence in X_{α,p_n}, (G_i) an admissible sequence of blocks such that $\{j : u_i(j) \neq 0\} \subset G_i$ and $s(u_i) \to 0$. Then a subsequence (v_i) of (u_i) satisfying*

$$\| \sum_{i=1}^{k} t_i v_i \, \|^{p_{n-1}} \leq 2(3)^{p_{n-1}-1} \sum_{i=1}^{k} \mid t_i \mid^{p_{n-1}} .$$

Lemma 2.3. *Let E_0 be an infinite dimensional subspace of $Z_p, j, n \in \mathbb{N}$ and $\varepsilon > 0$. There exist an $x \in E_0, x \neq 0$ and a $u \in Z_p$ such that*

(*i*) $||Q_n u|| \geq (1 - \varepsilon)||u||$

(*ii*) $||x - u|| < \varepsilon||u||$.

Lemma 2.4. *Suppose $\varepsilon > 0$ and ε_s for $s \in \mathbb{N}$ are such that*

$2\varepsilon_s \leq \varepsilon$ if $p = 1$,

$\sum_{s=1}^{\infty} (2\varepsilon_s)^q \leq \varepsilon^q$ if $1 < p < \infty$ where $\frac{1}{p} + \frac{1}{q} = 1$,

$\sum_{s=1}^{\infty} 2\varepsilon_s \leq \varepsilon$ if $p = 0$.

If, for given vectors$\{u_s\}_{s=1}^{\infty} \subset S(Z_p)$ *, there is a sequence of integers* $1 \leq n_1 < n_2 < ...$ *such that , for each* $s \in \mathbb{N}$ one has $\|u_s - Q_{n_s} u_s\| \leq \varepsilon_s$ and $\|Q_{n_{s+1}} u_s\| \leq \varepsilon_s$.

then $\{u_s\}_{s=1}^{\infty} \subset S(Z_p)$ is $(1 + \varepsilon)(1 - 3\varepsilon)^{-1}$ - equivalent to the unit vector basis of ℓ_p (respectively, c_0).

Theorem 2.5. *The Banach space Z_p is hereditarily ℓ_p if $1 \leq p < \infty$ and is hereditarily c_0 if $p = 0$.*

Lemma 2.6. *Z_1 fails the Schur property.*

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SUPER AMENABILITY AND BIPROJECTIVITY OF BEURLING ALGEBRAS

AMIN MAHMOODI

Abstract. We show that for each arbitrary weight function ω on a locally compact group G, the Beurling algebra $L^1(G,\omega)$ is super-amenable if and only if G is finite and it is biprojective if and only if G is compact.

1. Super-amenability

Let E be a Banach space. A *finite, biorthogonal system* for E is a set

$$\{(x_i,\varphi_j) : i,j = 1,...,n\},$$

where $x_1,...,x_n \in E$ and $\phi_1,...,\phi_n \in E^*$ satisfy

$$\langle x_i,\varphi_j\rangle = \delta_{i,j} \quad (i,j = 1,...,n).$$

The map $\theta_n : M_n \longrightarrow F(E)$ given by

$$\theta(A) := \sum_{i,j=1}^{n} a_{i,j}x_i \odot \varphi_j \quad (A = [a_{i,j}]_{i,j=1,...,n} \in M_n),$$

is a homomorphism, where the map $x_i \odot \varphi_j$ is defined by

$$x_i \odot \varphi_j : E \longrightarrow \mathcal{C}, x \longmapsto \langle x,\varphi_j\rangle x_i.$$

A Banach space E has the *property* **A** if there is a net of finite, bi-orthogonal systems

$$\{(x_i^{(\alpha)},\phi_j^{(\alpha)}) \quad : i,j = 1,...,n_\alpha\}$$

for E with corresponding homomorphisms $\theta_\alpha : M_{n_\alpha} \longrightarrow F(E)$ such that

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Key words and phrases. Super-amenability, biprojectivity, weighted group algebra, weight function.

$(i)\lim_\alpha \theta_\alpha(E_{n_\alpha}) = id_E$, uniformly on the compact subsets of E,
$(ii)\lim_\alpha \theta_\alpha(E_{n_\alpha})^* = id_{E^*}$, uniformly on the compact subsets of E^*, and
(iii) For each index α, there is a finite, irreducible $n_\alpha \times n_\alpha$ matrix group G_α such that $\sup_\alpha \sup_{g\in G_\alpha} \|\theta_\alpha(g)\| < \infty$.

For more details see [2, page 65]. A Banach algebra $\mathcal{A}$ is called *super-amenable* if for each Banach $\mathcal{A}$-bimodule E, each derivation $D : \mathcal{A} \longrightarrow E$ is inner. An element $\mathrm{m} \in \mathcal{A}\hat{\otimes}\mathcal{A}$ is called a *diagonal* for $\mathcal{A}$ if

$$a\Delta \mathrm{m} = a \quad , \quad a.\mathrm{m} = \mathrm{m}.a \quad (a \in \mathcal{A}),$$

where Δ is the *diagonal operator*

$$\Delta : \mathcal{A}\hat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}, a \otimes b \longmapsto ab.$$

$\mathcal{A}$ is super-amenable if and only if it has a diagonal, [5, Exercise 4.1.3]. Let G be a locally compact group with the left Haar measure λ and with identity e. A continuous map $\omega : G \longrightarrow \mathbb{R}^+$ is called a *weight function* on G if

$$\omega(xy) \le \omega(x)\omega(y),\ \omega(e) = 1,\ \omega(x) \ge 1 \quad (x, y \in G).$$

If ω is a weight function on G then the map

$$\omega \times \omega : G \times G \longrightarrow \mathbb{R}^+, (x, y) \longmapsto \omega(x)\omega(y)$$

is a weight function on $G \times G$. $L^1(G)$ has the property $\mathbf{A}$ [5, Exercise 3.1.4] and it is super-amenable if and only if G is finite [5, Exercise 4.1.7]. We have the same results for $L^1(G, \omega)$:

Theorem 1.1. *If G is a locally compact group and ω is a weight function on G , then $L^1(G, \omega)$ has the property* $\mathbf{A}$.

Proof. First suppose that the Haar measure of G is finite. Consider the collection of all families τ consisting of finitely many, pairwise disjoint sets in $\mathbf{B}_G$, the Borel algebra on G , such that $\lambda(A) \neq 0$ for each $A \in \tau$. For two such families τ_1 and τ_2 define $\tau_1 < \tau_2$ if each member of τ_1 is the union of a subfamily of τ_2 . For each $\tau = \{A_1, ..., A_{n_\tau}\}$ we have the corresponding finite, bi-orthogonal systems

$$\{(\frac{1}{\lambda(A_i)}\frac{\chi_{A_i}}{\omega}, \omega\chi_{A_j}) : i, j = 1, ..., n_\tau\}.$$

If $\theta_\tau : M_{n_\tau} \longrightarrow F(L^1(G, \omega))$ is the corresponding homomorphism, then

$$\theta_\tau(E_{n_\tau})(\frac{\chi_L}{\omega}) = \frac{\chi_L}{\omega}$$

and

$$\theta_\tau(E_{n_\tau})^*(\frac{\chi_L}{\omega}) = \frac{\chi_L}{\omega},$$

for each $L \in \mathbf{B}_G, L < \tau$. Thus $\lim_\alpha \theta_\alpha(E_{n_\alpha}) = id_E$ and $\lim_\alpha \theta_\alpha(E_{n_\alpha})^* = id_{E^*}$ uniformly on the compact subsets of E and E^*, respectively. Consider $\tau = \{A_1, ...A_{n_\tau}\}$ and let $\mathbf{G}_\tau$ be the group of matrices of the form $\mathbf{D_t E}_\sigma$, where $\mathbf{D_t}$ is the diagonal matrix specified by $\mathbf{t} = (t_i\delta_{i,j})$, where $t_1, ..., t_{n_\tau} \in \{-1, 1\}$, and $\mathbf{E}_\sigma$ is the matrix corresponding to the permutation σ of $\{1, ..., n_\tau\}$. Certainly $\mathbf{G}_\tau$ is an irreducible $n_\tau \times n_\tau$

matrix group. For each $f \in L^1(G,\omega)$ and $g = \mathbf{D_t E}_\sigma \in \mathbf{G}_\tau$ we have

$$\begin{aligned}\|\theta_\tau(\mathbf{D_t E}_\sigma)f\|_\omega &= \|\sum_{j=1}^{n_\tau}(\int_{A_j} f(t)\omega(t)d\lambda(t))\frac{1}{\lambda(A_\sigma(j))}\frac{\chi_{A_\sigma(j)}}{\omega}\|_\omega \\ &= \sum_{j=1}^{n_\tau}|t_j\int_{A_j} f(t)\omega(t)d\lambda(t)| \leq \sum_{j=1}^{n_\tau}\int_{A_j}|f(t)|\omega(t)d\lambda(t)| \leq \|f\|_\omega.\end{aligned}$$

Thus $\|\theta_\tau(g)\| \leq 1$. Finally in general case, we approximate the Haar measure λ with finite measures. □

Theorem 1.2. *$L^1(G,\omega)$ is super-amenable if and only if G is finite.*

Proof. If G is finite and of order n, then $m := \frac{1}{n}\sum_{g\in G}\delta_g \otimes \delta_{g^{-1}}$ is a diagonal for $L^1(G,\omega)$, and therefore $L^1(G,\omega)$ is super-amenable. Now let $L^1(G,\omega)$ is super-amenable, since it has the property **A** by Theorem 1.1, it has bounded approximation property. By [5, Theorem 4.1.5] there are $n_1, ..., n_k \in \mathbb{N}$ such that $L^1(G,\omega) \simeq M_{n_1} \oplus ... \oplus M_{n_k}$. Thus $L^1(G,\omega)$ has finite dimension and since it is unital [5, Exercise 4.1.1] so G is finite. □

2. Biprojectivity

A Banach algebra $\mathcal{A}$ is *biprojective* if the diagonal operator Δ has a bounded right inverse which is an $\mathcal{A}$-bimodule homomorphism. Similar to [1, Proposition 3.3.20] we have the following lemma.

Lemma 2.1. *Let G be a locally compact group and ω is a weight function on G. Then there is an isometric isomorphism $T : L^1(G,\omega)\hat{\otimes}L^1(G,\omega) \longrightarrow L^1(G\times G,\omega\times\omega)$ such that*

$$T(f \otimes g)(x,y) := f(x)g(y) \quad (f,g \in L^1(G,\omega),(x,y) \in G\times G). \quad (1)$$

Since G is compact if and only if $\lambda(G) < \infty$, we have the following lemma.

Lemma 2.2. *If ω is a weight function on G, then G is compact if and only if $\omega \in L^1(G)$.*

The map

$$\varphi_0 : L^1(G,\omega) \longrightarrow \mathcal{C}, f \longmapsto \int_G f(x)\omega(x)d\lambda(x)$$

is called *the augmentation* character on $L^1(G,\omega)$ and its kernel $L^1_0(G,\omega)$ is called the augmentation ideal of $L^1(G,\omega)$. It is a closed ideal of $L^1(G,\omega)$ with codimension one. Also $L^1_0(G,\omega)$ is *essential* as a left Banach $L^1(G,\omega)$-module, that is the linear hull of $\{g \star f : g \in L^1(G,\omega), f \in L^1_0(G,\omega)\}$ is dense in $L^1_0(G,\omega)$.

Theorem 2.3. *If ω is a weight function on G , then $L^1(G,\omega)$ is biprojective if and only if G is compact.*

Proof. Let G be a compact group and define the map

$$\rho : L^1(G,\omega) \longrightarrow L^1(G\times G,\omega\times\omega), \rho(f)(x,y) := f(xy) \quad (f \in L^1(G,\omega), x,y \in G).$$

We have $\Delta(f_1 \otimes f_2) = \int_G f_1 \otimes f_2(xy^{-1},y)d\lambda(y)$ for each f_1 and f_2 in $L^1(G,\omega)$ and $x \in G$. So $\Delta(F)(x) = \int_G F(xy^{-1},y)d\lambda(y)$ for each $F \in L^1(G\times G,\omega\times\omega)$ and $x \in G$. If

$f \in L^1(G,\omega)$ and $x \in G$, then $(\Delta\rho)(f)(x) = \int_G \rho(f)(xy^{-1}, y)d\lambda(y) = \int_G f(x)d\lambda(y) = f(x)$. Thus $\Delta\rho = id_{L^1(G,\omega)}$ and ρ is a $L^1(G,\omega)$-bimodule homomorphism and therefore $L^1(G,\omega)$ is biprojective. Let $\mathcal{A} := L^1(G,\omega)$ and $\mathbf{L} := L^1_0(G,\omega)$. By [Run, Lemma 4.3.10], the module map

$$\Theta : \mathcal{A}\hat{\otimes}\frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \frac{\mathcal{A}}{\mathbf{L}},\ f \otimes g + \mathbf{L} \longmapsto f \star g + \mathbf{L}$$

has a bounded right inverse ρ_1 which is also a left-$\mathcal{A}$-module homomorphism. By [5, Exercise 5.1.2 and Proposition 5.1.6], $\frac{\mathcal{A}}{\mathbf{L}}$ is *projective* and there is a left $\mathcal{A}$-module homomorphism $\tilde{\rho} : \frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \mathcal{A}$ such that $\pi\tilde{\rho} = id_{\frac{\mathcal{A}}{\mathbf{L}}}$, where $\pi : \mathcal{A} \longrightarrow \frac{\mathcal{A}}{\mathbf{L}}$ is canonical epimorphism.

The map

$$\tilde{\phi_0} : \frac{\mathcal{A}}{\mathbf{L}} \longrightarrow \mathbb{C},\ f + \mathbf{L} \longmapsto \int_G f(x)d\lambda(x)$$

is an isomorphism. Now set $\rho := \tilde{\rho}\tilde{\phi_0}^{-1}$ and $f_0 := \rho(1) \in \mathcal{A}$. We have

$$\phi_0(f_0) = \phi_0(\rho(1)) = \phi_0(\tilde{\rho}\tilde{\phi_0}^{-1})(1) = 1.$$

$\mathbb{C}$ is a left Banach $\mathcal{A}$-module with the module action $f.\alpha := \phi_0(f)\alpha \quad (\alpha \in \mathbb{C}, f \in \mathcal{A})$. Since ρ is a left $\mathcal{A}$-module homomorphism, for each $f \in \mathcal{A}$ we have

$$f \star f_0 = f \star \rho(1) = \rho(f.1) = \rho(\phi_0(f)1) = \phi_0(f)\rho(1) = \phi_0(f)f_0,$$

and for each $x \in G$ and $f \in \mathcal{A} - \mathbf{L}$

$$\phi_0(f)f_0 = \phi_0(\delta_x \star f)f_0 = (\delta_x \star f) \star f_0 = \delta_x \star (f \star f_0) = \delta_x \star (\phi_0(f)f_0) = \phi_0(f)(\delta_x \star f_0).$$

Thus f_0 is non-zero, constant function in $\mathcal{A}$, and by Lemma 2.2, G is compact. □

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QUASICOMPACT AND RIESZ ENDOMORPHISMS OF INFINITELY DIFFERENTIABLE LIPSCHITZ ALGEBRAS

HAKIMEH MAHYAR

Abstract. In this note we study the endomorphisms of the infinitely differentiable Lipschitz algebras $\mathrm{Lip}(X, M, \alpha)$, which are quasicompact operators or Riesz operators. We show that under certain conditions every quasicompact or Riesz endomorphism of these algebras is necessarily power compact. Then, when $\mathrm{Lip}(X, M, \alpha)$ is a natural Banach function algebra, we determine the spectra of Riesz and quasicompact endomorphisms of this algebra.

1. Introduction and preliminaries

Let X be a perfect compact plane set, and $0 < \alpha \leq 1$. The Lipschitz algebra $\mathrm{Lip}(X, \alpha)$ of order α, is the algebra of all complex-valued functions f on X for which

$$p_\alpha(f) = \sup\{\frac{|f(z) - f(w)|}{|z - w|^\alpha} : z, w \in X, z \neq w\} < \infty.$$

The algebra $\mathrm{Lip}(X, \alpha)$ is a Banach function algebra on X, if equipped with the norm $\|f\|_\alpha = \|f\|_X + p_\alpha(f)$, where $\|f\|_X = \sup_{x\in X} |f(x)|$. The complex-valued function f on X is called complex-differentiable on X if at each point $z_0 \in X$ the limit

$$f'(z_0) = \lim_{\substack{z \to z_0 \\ z \in X}} \frac{f(z) - f(z_0)}{z - z_0}$$

exists. The algebra of functions f on X whose derivatives of all orders exist and $f^{(n)} \in \mathrm{Lip}(X, \alpha)$ for all n, is denoted by $\mathrm{Lip}^\infty(X, \alpha)$.

We now introduce certain subalgebras of $\mathrm{Lip}^\infty(X, \alpha)$. Let $M = (M_n)$ be a sequence of positive numbers satisfying $M_0 = 1$ and $\frac{M_{n+m}}{M_n M_m} \geq \binom{n+m}{n}$, where m and n are nonnegative integers. We define infinitely differentiable Lipschitz algebras $\mathrm{Lip}(X, M, \alpha)$,

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which were first studied in [2], [3], as follows:

$$\mathrm{Lip}(X, M, \alpha) = \{f \in \mathrm{Lip}^\infty(X, \alpha) : \|f\| = \sum_{k=0}^{\infty} \frac{\|f^{(k)}\|_\alpha}{M_k} < \infty\}.$$

With the pointwise addition and multiplication, $\mathrm{Lip}(X, M, \alpha)$ is a commutative normed algebra which is not necessarily complete. However for certain compact plane sets X, the algebras $\mathrm{Lip}(X, M, \alpha)$ are Banach function algebras on X. For example, if the compact plane set X satisfies the following condition which is called the $(*)$-condition:

$(*)$ *There exists a constant C such that for every $z, w \in X$ and every continuously differentiable function f on X,*

$$|f(z) - f(w)| \leq C|z - w|(\|f\|_X + \|f'\|_X).$$

Such sets include the closed interval, the circle, the annulus and the closed unit disc. Moreover, for certain compact plane sets X, when $M = (M_n)$ is a non-analytic sequence, i.e. $\lim_{n\to\infty}(\frac{n!}{M_n})^{1/n} = 0$, and $0 < \alpha < 1$, the algebras $\mathrm{Lip}(X, M, \alpha)$ are natural, i.e. their maximal ideal spaces are X.

The compact endomorphisms of $\mathrm{Lip}(X, M, \alpha)$ were studied in [4], [5]. In this note we study quasicompact or Riesz endomorphisms of these algebras.

If E is a Banach space, we denote by $\mathcal{B}(E)$ the Banach space of all bounded operators on E, and by $\mathcal{K}(E)$ the Banach space of all compact operators on E. The *essential spectral radius* $\rho_e(T)$ of $T \in \mathcal{B}(E)$, is given by the formula

$$\rho_e(T) = \lim_{n\to\infty} \|T^n + \mathcal{K}(E)\|^{\frac{1}{n}} = \lim_{n\to\infty} (dist(T^n, \mathcal{K}(E)))^{\frac{1}{n}}.$$

We say that T is a *Riesz operator* if $\rho_e(T) = 0$ and *quasicompact* if $\rho_e(T) < 1$. An operator T on a Banach space is said to be *power compact* if there exists a positive integer N such that T^N is compact. Obviously, every power compact operator is Riesz and hence quasicompact. Certainly, in general the converse is not true. In this note we show that in the algebra $\mathrm{Lip}(X, M, \alpha)$, every quasicompact endomorphism and so every Riesz endomorphism is necessarily power compact.

In general, if T is a unital endomorphism of a unital commutative semi-simple Banach algebra B with the maximal ideal space $\mathcal{M}(B)$, then there exists a w*-continuous selfmap $\varphi : \mathcal{M}(B) \to \mathcal{M}(B)$ such that $\widehat{Tf} = \hat{f} \circ \varphi$ for all $f \in B$. In fact, φ is equal to the adjoint T^* restricted to $\mathcal{M}(B)$. In this case we say φ induces T. If a Banach function algebra B on a compact Hausdorff space X is natural, then every nonzero endomorphism T of B has the form $Tf = f \circ \varphi$ for a selfmap φ of X.

2. Riesz and quasicompact endomorphisms

Let X be a perfect compact plane set, let $M = (M_n)$ be a non-analytic weight sequence and let $0 < \alpha \leq 1$. We say an infinitely differentiable selfmap φ of X is analytic if

$$\sup_n \left(\frac{\|\varphi^{(n)}\|_X}{n!} \right)^{1/n} < \infty.$$

Using [4, Theorem 4.4] and [1, Theorem 1.2(ii)], we show that if the fixed point of φ is in the interior of X then every quasicompact endomorphism of $\mathrm{Lip}(X, M, \alpha)$ induced by φ, is power compact.

Theorem 2.1. Let X be a connected compact plane set, $0 < \alpha \leq 1$, $M = (M_n)$ be a non-analytic weight sequence and the Banach function algebra $\mathrm{Lip}(X, M, \alpha)$ be natural. Suppose that T is a quasicompact endomorphism of $\mathrm{Lip}(X, M, \alpha)$ induced by the analytic selfmap φ of X. If φ has an interior fixed point z_0, then T is a power compact.

Also by using [1, Theorem 1.2(ii)] and [4, Theorems 3.2 and 4.2], we show that every quasicompact endomorphism of $\mathrm{Lip}(X, M, \alpha)$ is power compact, which is a more general result than Theorem 2.1.

Theorem 2.2. Let X be a connected compact plane set satisfying the $(*)$-condition, $0 < \alpha \leq 1$ and $M = (M_n)$ be a non-analytic weight sequence such that the Banach function algebra $\mathrm{Lip}(X, M, \alpha)$ is natural. Then every quasicompact endomorphism T induced by an analytic selfmap φ of X is power compact.

3. Spectra of Riesz and quasicompact endomorphisms

In this section we determine the spectrum $\sigma(T)$ of a Riesz and quasicompact endomorphism T of a Banach function algebra $\mathrm{Lip}(X, M, \alpha)$.

Theorem 3.1. Let X be a perfect compact plane set with nonempty interior such that the Banach function algebra $\mathrm{Lip}(X, M, \alpha)$ is natural. Let T be a Riesz endomorphism of $\mathrm{Lip}(X, M, \alpha)$ induced by a selfmap φ. If φ has an interior fixed point z_0, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbb{N}\} \cup \{0, 1\}$.

In the above results we need the underlying set X to have a nonempty interior. In the following we determine the spectrum of a Riesz endomorphism of $\mathrm{Lip}(X, M, \alpha)$ for uniformly regular sets X without assuming a nonempty interior for X. First, in general, we have

Theorem 3.2. Let B be a natural Banach function algebra on a perfect compact plane set X containing the coordinate function z and consisting of continuously differentiable functions on X. If a selfmap φ induces a Riesz endomorphism T of B and z_0 is a fixed point of φ, then $\{\varphi'(z_0)^n : n \in \mathbb{N}\} \cup \{0, 1\} \subseteq \sigma(T)$.

Theorem 3.3. Let X be a uniformly regular set, $0 < \alpha \leq 1$, $M = (M_n)$ a non-analytic weight sequence, and $\mathrm{Lip}(X, M, \alpha)$ be natural. Suppose a selfmap φ induces a Riesz endomorphism T of $\mathrm{Lip}(X, M, \alpha)$ and z_0 is the fixed point of φ. If either φ is analytic or there exists a neighbourhood U of z_0 such that $U \cap X$ is convex, then $\sigma(T) = \{\varphi'(z_0)^n : n \in \mathbb{N}\} \cup \{0, 1\}$.

Remark 3.1. If the inducing map φ is analytic, the results of Theorems 3.1 and 3.3 hold for quasicompact endomorphisms.

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MAJORIZATION IN SEMIFINITE VON NEUMANN ALGEBRAS

SEYED MAHMOUD MANJEGANI

ABSTRACT. In the theory of matrices, several majorization are known for the eigenvalues and the singular values of matrices, which are useful in deriving norm inequalities for matrices. In this talk we introduce the concept of majorization and submajorization with their several characterization in semifinite Von Neumann algebra $\mathcal{M}$ with a faithful normal trace $\mathcal{T}$.

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ON BEST COAPPROXIMATION IN L^1

HAMID MAZAHERI

Abstract. As a counterpart to best approximation in normed linear spaces, best coapproximation was introduced by Franchetti and Furi. In this paper, we shall show that if M is an approximatively compact subspace of X and M or $\breve{M}$ is separable, then $L^1(S, M)$ is proximinal in $L^1(S, X)$.

1. Introduction and preliminaries

Let X be a normed linear space and M be a nonempty subspace of X. Then a point $g_0 \in M$ is said to be a best coapproximation for $x \in X$ if for every $g \in M$,

$$\|g - g_0\| \leq \|x - g\|.$$

If each $x \in X$ has at least one best coapproximation in M, then M is called a coproximinal subspace of X. If M is a coproximinal subspace in X, then M is closed in X. If each $x \in X$ has a unique best coapproximation in M, then M is called a cochebyshev subspace of X.
Let M be a subspace of a normed linear space X, then for $x \in X$ we put

$$R_M(x) = \{g_0 \in M : \|g - g_0\| \leq \|x - g\| \ \forall g \in M\}$$

the set of all best coapproximations for x in M. It is clear that $R_M(x)$ is closed, bounded and convex subset of X. The set-valued function R_M which associate to each x in X, the set of all its best coapproximations is called cometric projection opeator. Put

$$\breve{M} = \{x \in X \ : \ \|g\| \leq \|g - x\| \ \forall g \in M\},$$

and

$$\widehat{M} = \{x \in X \ : \ \|x\| \leq \|g - x\| \ \forall g \in M\}.$$

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A subset M of a Banach space X is called approximatively compact if for every $x \in X$ and every sequence $\{g_n\}_{n\geq 1}$ in M with $\lim_{n\to\infty} \|x-g_n\| = \|x-M\|$, there exists a subsequence $\{g_{n_k}\}_{k\geq 1}$ converging to an element of M.

Let X be a Banach space and (S,M,μ) be a finite complete measure space. A function $\varphi : S \to X$ is said to be simple if its range contains only finitely many points $x_1, x_2, \ldots, x_n \in X$, and if $\varphi^{-1}(x_i)$ is measurable for all $i = 1, 2, \ldots, n$. Such φ can be written as $\varphi = \sum_{i=1}^{n} x_i \chi_{E_i}$, where χ_{E_i} is the characteristic function of the set $E_i = \varphi^{-1}(x_i)$. A function $f : S \to X$ is said to be strongly measurable if there exists a sequence $\{\varphi_n\}$ of simple functions with $\lim_{n\to\infty} \|\varphi_n(t) - f(t)\| = 0$ almost everywhere $[d\mu]$.

The space of Bochner p-integrable functions is denoted by $L^p(S,X)$ which contains of all strongly measurable functions $f : S \to X$ such that

$$\int_S \|f(t)\|^p d\mu(t) < \infty \quad , \quad 1 \leq p < \infty \cdot$$

The norm in $L^p(S,X)$ is defined to be $\|f\|_p = (\int_S \|f(t)\|^p d\mu(t))^{\frac{1}{p}}$. It is known that $L^p(S,X)$ is a Banach space.

The problems of coapproximation was initially introduced by Franchetti and Furi ,[1], in order to study some characteristic properties of real Hilbert spaces, and such problems were considered further by Papini and Singer [5]. Also there are some results on coapproximation in [2-3]. In this context, we shall consider coproximinality in L^1.

We conclude this section by a list of know facts needed in the proof of the main results.

Lemma 1.1. *Let M be a closed subspace of X. Then M is coproximinal in X if and only if*

$$X = M + \breve{M} = \{g + \breve{g} : g \in M, \breve{g} \in \breve{M}\}.$$

Lemma 1.2. *Every non-empty approximatively compact set M in X is proximinal and in this case P_M is upper semi-continuous.*

Lemma 1.3. *If $f : S \longrightarrow X$ is measurable in the classical sense and has essentially separable range, then f is strongly measurable.*

Lemma 1.4. *Let $\varphi : S \longrightarrow 2^Y$ be a weakly measurable set valued function. If Y is a complete separable metric space, then φ has a measurable selection.*

Lemma 1.5. *Let $\varphi : S \longrightarrow 2^Y$ be a set valued function. If Y is a complete separable metric space, then φ is weakly measurable if and only if $\varphi^{\omega}(C)$ is a measurable subset of S for every closed set C of Y.*

Lemma 1.6. *Let M be a linear subspace of $X, x \in X\backslash\overline{M}$ and $g_0 \in M$. Then $g_0 \in R_M(x)$ if and only if For all $g \in M$ there exists $f^g \in X^*$ such that $\|f^g\| = 1$, $f^g(x - g_0) = 0$ and $f^g(g) = \|g\|$.*

2. Main results

Let X and Y be normed spaces and $R_c(Y)$ denotes the set of all nonempty and closed subsets of Y. A mapping $\psi : X \longrightarrow R_c(Y)$ is called upper semi-continuous if the set

$$\{x \in X : \psi(x) \subseteq V\}$$

is open for every open set V in Y, or equivalently, the set

$$\{x \in X : \psi(x) \cap N \neq \emptyset\}$$

is closed for every closed set N in Y.

Theorem 2.1. *Let X be a normed space, and M be a coproximinal subspace of X. Then R_M is upper semi-continuous if and only if $F + \breve{M}$ is closed for every closed set F in M.*

Theorem 2.2. *Each of the following two conditions implies that π is weakly measurable..*

a) M be a coproximinal separable subspace of X and R_M be upper semi-continuous,

b) M be a coproximinal separable subspace of X. Let $\breve{M}$ be convex approximatively compact in X.

Theorem 2.3. *Suppose M is coproximinal and π is weakly measurable. If M is separable, then $L^1(S, M)$ is proximinal in $L^1(S, X)$.*

Corollary 2.4. *If M is any closed subspace of a Hilbert space H. Then $L^1(S, M)$ is coproximinal in $L^1(S, H)$.*

Lemma 2.5. *Let G be a closed subspace of X. Then $g \in L^1(S, M)$ is a best coapproximation for an element f of $L^1(S, X)$ if and only if for almost all $s \in S$, $g(s)$ is a best coapproximation for $f(s)$.*

Lemma 2.6. *If $L^1(S, M)$ is a cochebyshev subspace of $L^1(S, X)$, then M is a cochebyshev subspace of X.*

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CHARACTERIZATIONS OF FRAGMENTABILITY AND σ-FRAGMENTABILITY

A. KAMEL MIRMOSTAFAEE [1*] AND M. MIRZAVAZIRI [2]

Abstract. The notion of fragmentability of a topological space is appeared in a paper of Jayne and Rogers. It turned out that this notion have interesting connection with other properties of Banach spaces, and attracted the attention of recent research workers. In this talk, we discuss about characterizations of fragmentability and σ-fragmentability, which are obtained by N. Ribarska, kenderov and Moors. This notions have close connection with other topics in Banach spaces such as analytic spaces, the Radon-Nikodym property, differentiability of convex functions and Kadec renorming.

1. Introduction

Let (X, τ) be a topological space and let ρ be a pseudo-metric on X. Given $\epsilon > 0$, a non empty subset A of X is said to be fragmented by ρ down to ϵ if each non-empty subset B of A has a relatively τ-open subset of diameter less than ϵ. The set A is said to be fragmented by ρ, if A is fragmented by ρ down to ϵ for each $\epsilon > 0$. The set A is said to be σ-fragmented by ρ, if for each $\epsilon > 0$, A can be expressed as $A = \bigcup_{n=1}^{\infty} A_n$ with each A_n fragmented by ρ down to ϵ. The notion of fragmentability and its variants have been found to have interesting connection with other topics such as locally uniformly convex and Kadec renorming of Banach spaces and the differentiability of convex function on Banach spaces.
In this paper, characterizations of fragmentability and σ-fragmentability, which are obtained by Ribarska [3] and kenderov-Moors [2], will be investigated.

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* Speaker.

2. Relatively open partitioning and fragmentability

A well ordered family $\mathcal{U} = \{U_\zeta : 0 \leq \zeta < \zeta_0\}$ of subsets of the topological space X is said to be a *relatively open partitioning* of X if
(i)$U_0 = \emptyset$;
(ii) U_ζ is contained in $X\backslash(\cup_{\eta<\zeta} U_\eta)$ and is relatively open in it for every $\zeta, 0 < \zeta < \zeta_0$;
(iii)$X = \bigcup_{\zeta<\zeta_0} U_\zeta$.
A family $\mathcal{U}$ of subsets of the topological space X is said to be a $\sigma-$*relatively open partitioning* of X, if $\mathcal{U} = \bigcup_{n=1}^{\infty} \mathcal{U}^n$ and $\mathcal{U}^n$, $n = 1, 2, \ldots$, are relatively open partitioning of X. $\mathcal{U}$ is said to separate the points of X, if whenever x and y are two different elements of X, there exists n such that x and y belong to different elements of the partitioning $\mathcal{U}^n$.In this case we say that X admits a separating $\sigma-$relatively open partitioning.

The relation between fragmentability and σ-relatively open partitioning is described in the following theorem:

Theorem 2.1. ([4], theorem 1.9) *The topological space X admits a separating $\sigma-$relatively open partitioning , if and only if there exists a metric which fragments X.*

We could consider topological spaces which are fragmented by some non-negative function $\lambda : X \times X \to [0, \infty)$ with $\lambda(x, y) = 0$ if and only if $x = y$. Ribarska noted that if there exists such a fragmenting function on X, then there exists a metric which fragments it [3].
The following corollary is obtained by Ribarska by means of theorem 2.1 (cf [3], p.247]):

Corollary 2.2. *Let X be a Hausdorff compact space, which admits separating $\sigma-$relatively open partionning. Then there exists a complete metric ρ on X, which fragments it and such that the topology of ρ is stronger than the original topology on X.*

Ribarska used the notion of σ-relatively open partitioning to prove the following result, which gives us a large class of fragmentable spaces [3]:

Theorem 2.3. *Let X be a Hausdorff compact space which is fragmented by a metric. Then the space $C(X)^*$, endowed with the weak star topology is a fragmented space as well.*

The class of fragmentable spaces is closed under taking subspaces, countable product and perfect images, more precisely, we have:

Proposition 2.4. ([3] p.250) *Let $\mathcal{M}$ be the class of fragmentable spaces.*
(a)If $X \in \mathcal{M}$ then $X_1 \in \mathcal{M}$ for every $X_1 \subset X$.
(b)Let $X \in \mathcal{M}$ and g be a perfect mapping from X onto the topological space Y (i.e. g is a continuous mapping which maps closed subsets of X into closed subsets of Y and $g^{-1}(y)$ is a compact non-empty subset of X for every $y \in Y$), then $Y \in \mathcal{M}$.)

(c)From $X = \bigcup_{i=1}^{\infty} X_i$ where all X_i are closed in X and $X_i \in \mathcal{M}, i = 1, 2 \ldots,$ it follows that $X \in \mathcal{M}$.
(d) If the spaces $X_i, i = 1, 2, \ldots$ are in $\mathcal{M}$, then their cartesian product $\Pi_{i=1}^{\infty} X_i$ also belongs to $\mathcal{M}$.
(e) Let $X \in \mathcal{M}$ and h be a continuous one-one mapping from Y into X. Then Y is in $\mathcal{M}$.

3. Game characterization of fragmentability

In [1] the following topological game was used to characterize fragmentability of the space X. Two players Σ and Ω select alternatively subsets of X. Σ starts the game by selecting an arbitrary non-empty subset A_1 of X. Then Ω chooses some nonempty subset B_1 of A_1 which is relatively open in A_1. In general, if the selection $B_n \neq \emptyset$ of the player Ω is already specified, the player Σ makes the next move by selecting an arbitrary nonempty set $A_{n+1} \subset B_n$. In return, Ω selects a nonempty relatively open subset B_{n+1} of A_{n+1}. Continuing this alternative selection of sets in X a sequence $A_1 \supset B_1 \supset A_2 \supset \ldots A_n \supset B_n \supset \ldots$ is generated which we call a *play* and denote by $p = (A_i, B_i)_i$. The player Ω is said to have won the play $p = (A_i, B_i)_i$ if the set $\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i$ contains at most one point. Otherwise the player Σ is said to have won the play $p = (A_i, B_i)_i$.
This game will be denoted by $p = (A_1, B_1, \cdots, A_n, B_n, \cdots)$. A finite stage of the game , when Ω is to move, we will call a *partial play* and denote by

$$p_n = (A_1, B_1, \cdots, A_n).$$

A *strategy* ω for the player Ω is a mapping that assigns to each partial play p_n some nonempty relatively open subset $\omega(A_1, B_1, \ldots, A_n)$ of A_n. Given the strategy ω, we call the play $p = (A_i, B_i)_i$ an ω-*play* if $B_i = \omega(A_1, B_1, \ldots, A_i)$ for every $i \geq 1$. i.e. p is an ω-play if Ω makes his/her selections by means of (or according to) the strategy ω. The strategy ω is called *winning strategy* for Ω if every ω-play is won by the player Ω. If the space X is fragmentable by the metric $d(.,.)$, then Ω has an obvious winning strategy ω. Indeed, to each partial play $A_1 \supset B_1 \supset \ldots \supset A_k$ this strategy puts into correspondence some nonempty subset $B_n \subset A_n$ which is relatively open in A_n and has d-diameter less than $1/n$. Clearly, the set $\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i$ has at most one point because it has d-diameter 0. It turns out the existence of winning strategy for the player Ω characterizes the fragmentability of Ω.

Theorem 3.1. ([1], theorem 1.1) *The topological space X is fragmentable if, and only if, the player Ω has a winning strategy for the above game .*

If $t_1,\ t_2$ are two (not necessarily distinct) topologies on the set X, when is (X, t_1) fragmentable by a metric $d(.,.)$ such that the topology generated by $d(.,.)$ is stronger than or equal to the topology t_2? The basic example we have in mind is the case when X is a Banach space, t_1 is the weak topology and t_2 is the norm topology. The following statement gives an answer to this question [2]:

Theorem 3.2. *Let $t_1,\ t_2$ be two (not necessarily distinct) topologies on the set X. The space (X, t_1) is fragmentable by a metric $d(.,.)$ which majorizes t_2 if, and only if, there exists a strategy ω for the player Ω in the game G in (X, t_1) such that, for every ω-play $p = (A_i, B_i)_i$, either $\bigcap_{i \geq 1} A_i = \bigcap_{i \geq 1} B_i = \emptyset$ or $\bigcap_{i \geq 1} B_i = \{x\}$ for some $x \in X$ and, for every t_2-open set $U \ni x$, there exists some integer $k > 0$ with $B_k \subset U$.*

The relation between fragmentability and $\sigma-$fragmentability is shown in the following result due to Ribarska [4].

Theorem 3.3. *Let (X, τ) be a topological space and ρ be a lower semi-continuous metric which σ fragments it. Then (X, τ) is a fragmentable space.*

Since the norm is lower semi-continuous with respect to weak topology, the above statement implies that (X,weak)is fragmented by the norm if it is $\sigma-$fragmentable. In Banach spaces, the picture is more interesting,that is [2]:

Theorem 3.4. *For a Banach space X the following assertions are equivalent:*
(a)$(X, weak)$ is fragmentable by a metric $d(.,.)$ which majorizes the weak topology;
(b)$(X, weak)$ is fragmentable by a metric $d(.,.)$ which majorizes the norm topology;
(c) $(X, weak)$ is σ-fragmentable by a metric $d(.,.)$ which majorizes the weak topology;
(d)$(X, weak)$ is σ-fragmentable by the norm.

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FIXED POINT THEOREM FOR G-CONVEX SPACE AND ARC-LIKE PLANE CONTINUUM

MONA NABIEI

Abstract. In this paper,we study the generalization of convex space and nonseparating plane continuum, and obtain a version of fixed point theorem in this space.

1. Introduction and preliminaries

In the first half of the twentieth century, when foundations of general topology had been established, many famous topologists were particularly interested in properties of compact connected metric spaces, called continua. What later emerged as continuum theory is a continuation of study. Continuum theory not exactly is a "theory" separated from other areas of topology and mathematics, and its identity is rather defined by special type of questions asked in this area. When basic general topology is already established, many deep but naturally and simply formulated problems in continuum theory still remain open. Due to those problems, continuum theory remains remarkably fresh among other areas of topology. C. L. Hagopian collected some open problem in continuum theory, which one of them is about nonseparating plane continuum[2]. In this paper, we obtain a fixed point theorem for nonseperating plane continuum, and a same way to establish this theorem in generalized convex (or G-convex) spaces, which has been introduced by S. Park in 1993 [4]. See also [1, 2, 3]

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AUTOMATIC CONTINUITY IN TOPOLOGICAL Q-ALGEBRAS

MASOUMEH NAJAFI TAVANI

Abstract. We first show that in a topological Q-algebra, the spectrum function is upper semicontinuous and hence the spectral radius function is upper semicontinuous. Then we prove that each homomorphism from a topological Q-algebra onto an advertibly complete semisimple lmc algebra has a closed graph. Consequently, if A is a complete metrizable Q-algebra and B is a semisimple Fréchet algebra, then every surjective homomorphism $T : A \longrightarrow B$ is automatically continuous.

1. Introduction and preliminaries

A short proof of Johnson's uniqueness of norm theorem has been presented by Ransford [5]. Later on, by following the technique of Ransford, Fragoulopoulou proved that if A and B are lmc algebras such that A is a Q-algebra, B is semisimple and advertibly complete and (A, B) is a closed graph pair, then each surjective homomorphism $\phi : A \longrightarrow B$ is continuous [3, Theorem 2.6]. Using this result, she then concluded that some classes of semisimple lmc algebras, such as LFQ-algebras and Fréchet Q-algebras have a uniquely determined topology [3]. In 1996 Akkar and Nacir obtained the same result for LFQ-algebras [1, Theorem 4] by using the fact that the spectral radius function $x \mapsto \rho_A(x)$, is upper semicontinuous on each topological Q-algebra A [1, Lemma 1]. In this work we first show that for a topological Q-algebra A, the spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous. This clearly implies the upper semicontinuity of the spectral radius function $x \mapsto \rho_A(x)$. Then we generalize some of the above results by removing the lmc property.

We now present some definitions and known results. For further details one can refer, for example, to [2].

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Key words and phrases. Topological algebra, Q-algebra, spectrum function, upper semicontinuity, automatic continuity.

Definition 1.1. A locally multiplicatively convex (lmc) algebra is a topological algebra whose topology is defined by a separating family $\mathcal{P} = (p_\alpha)$ of submultiplicative seminorms. A complete metrizable lmc algebra is a Fréchet algebra.

The topology of a Fréchet algebra can be defined by an increasing sequence (p_n) of submultiplicative seminorms.

Definition 1.2. An F-algebra is a topological algebra whose underlying topological linear space is an F-space, or in other words, the topology of an F-algebra is defined by a complete invariant metric.

Note that a Fréchet algebra is an F-algebra which is also an lmc algebra.

Definition 1.3. A topological algebra A is a Q-algebra if the set of all quasi invertible elements of A $(q - InvA)$ is open in A.

If A is unital then it is easy to see that A is a Q-algebra if and only if $InvA$, the set of all invertible elements of A, is open.

Definition 1.4. A topological algebra A is advertibly complete if a Cauchy net (x_α) in A converges in A whenever for some $y \in A$, $x_\alpha + y - x_\alpha.y$ converges to zero.

Note that a topological Q-algebra is advertibly complete [4, p. 45].

For a unital topological algebra A let $sp_A(x)$ denote the spectrum of $x \in A$ and $\rho_A(x)$ denote the spectral radius of $x \in A$. We set $\rho_A(x) = +\infty$ if $sp_A(x)$ is unbounded and $\rho_A(x) = 0$ if $sp_A(x) = \emptyset$.

Remark 1.5. Let A be an lmc algebra with the family of seminorms $\mathcal{P} = (p_\alpha)$. Let A_α denote the Banach algebra obtained by the completion of $A/\ker p_\alpha$ in the norm $p'_\alpha(x + \ker p_\alpha) = p_\alpha(x)$. Since $sp_{A_\alpha}(x + \ker p_\alpha) \subseteq sp_A(x)$ and $sp_{A_\alpha}(x + \ker p_\alpha) \neq \emptyset$, we have $sp_A(x) \neq \emptyset$. If, moreover, A is advertibly complete, then

$$\rho_A(x) = \sup_\alpha (\lim_{n\to\infty} (p_\alpha(x^n)^{\frac{1}{n}}) = \sup_\alpha \rho_{A_\alpha}(x + \ker p_\alpha)$$

[4, Theorem III, 6.1].

Definition 1.6. Let A be a topological algebra. The spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous at $a \in A$ if, for every open set U containing $sp_A(a)$, there exists a neighbourhood V of a such that $sp_A(x) \subseteq U$ whenever $x \in V$.

We need the following known results in the next section. See, for example, [2, Proposition 1.5.32 and Lemma 5.1.8] or [5].

Lemma 1.7. *If A is a unital algebra then*

$$radA = \{x \in A : \forall y \in A, \rho_A(xy) = 0\},$$

where $radA$ is the Jacobson radical of A.

Lemma 1.8. *Let A be a Banach algebra and let $p(z)$, for $z \in \mathbb{C}$, be a polynomial with coefficients in A. Then for each $R > 0$ we have*

$$\rho_A^2(p(1)) \leq \sup_{|z|=R} \rho_A(p(z)). \sup_{|z|=\frac{1}{R}} \rho_A(p(z))$$

.

2. Main results

In this section we assume that all algebras are unital with the unit e.

Theorem 2.1. *Let A be a topological algebra. If A is a Q-algebra then the spectrum function $x \mapsto sp_A(x)$ is upper semicontinuous on A.*

The above theorem clearly implies the upper semicontinuity of the spectral radius function, which will be used in the proof of theorem 2.3.

To prove the next result we need the following elementary lemma.

Lemma 2.2. *Let f be an upper semicontinuous real-valued function on a topological space X, and K be a compact subset of X. Then f takes its maximum on K.*

Theorem 2.3. *Let A be a topological Q-algebra and let B be an lmc semisimple algebra which is advertibly complete. If $T : A \longrightarrow B$ is a surjective homomorphism then T has a closed graph.*

Corollary 2.4. *Let A be an F-algebra which is also a Q-algebra and let B be a semisimple Fréchet algebra. Then every surjective homomorphism $T : A \longrightarrow B$ is automatically continuous.*

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STABILITY OF AN APOLLONIUS TYPE QUADRATIC MAPPING

ABBAS NAJATI

Abstract. Let X and Y be linear spaces. It is shown that for a fixed positive integer $n \geq 2$, if a mapping $Q : X \to Y$ satisfies the following functional equation

$$\sum_{i=1}^{n} Q(z - x_i) = \frac{1}{n} \sum_{\substack{1 \leq i,j \leq n \\ j<i}} Q(x_i - x_j) + nQ\left(z - \frac{1}{n}\sum_{i=1}^{n} x_i\right) \tag{0.1}$$

for all $z, x_1, \ldots, x_n \in X$, then the mapping $Q : X \to Y$ is a *quadratic mapping of Apollonius type* and a quadratic mapping. We moreover prove the Hyers–Ulam stability of the functional equation (0.1) in Banach spaces.

1. Introduction and preliminaries

The stability problem of functional equations originated from a question of S.M. Ulam [4] concerning the stability of group homomorphisms: Let G_1 be a group and let G_2 be a metric group with the metric $d(.,.)$. Given $\epsilon > 0$, does there exist a $\delta > 0$ such that if a function $h : G_1 \to G_2$ satisfies the inequality $d(h(xy), h(x)h(y)) < \delta$ for all $x, y \in G_1$, then there exists a homomorphism $H : G_1 \to G_2$ with $d(h(x), H(x)) < \epsilon$ for all $x \in G_1$?

In an inner product space, the equality

$$\|z - x\|^2 + \|z - y\|^2 = \frac{1}{2}\|x - y\|^2 + 2\left\|z - \frac{x + y}{2}\right\|^2 \tag{1.1}$$

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holds, and is called the *Apollonius' identity.* The following functional equation, which was motivated by this equation,

$$Q(z-x)+Q(z-y)=\frac{1}{2}Q(x-y)+2Q\Big(z-\frac{x+y}{2}\Big), \tag{1.2}$$

is quadratic (see [3]). For this reason, the functional equation (1.2) is called a *quadratic functional equation of Apollonius type,* and each solution of the functional equation (1.2) is said to be a *quadratic mapping of Apollonius type* [2, 3]. The quadratic functional equation and several other functional equations are useful to characterize inner product spaces [1].

In [3], C. Park and Th.M. Rassias introduced and investigated a functional equation, which is called the *generalized Apollonius type quadratic functional equation.*

In this note, employing the above equality (1.2), for a fixed positive integer $n \geq 2$, we introduce a new functional equation, which is called the *quadratic functional equation of* n*–Apollonius type* and whose solution of the functional equation is said to be a *quadratic mapping of* n*–Apollonius type,*

$$\sum_{i=1}^{n} Q(z-x_i)=\frac{1}{n}\sum_{\substack{1\leq i,j\leq n\\ j<i}} Q(x_i-x_j)+nQ\Big(z-\frac{1}{n}\sum_{i=1}^{n}x_i\Big). \tag{1.3}$$

We introduce the n–Apollonius' identity in an inner product space for a fixed positive integer $n \geq 2$. We show that the quadratic functional equation of n–Apollonius type (1.3) is a quadratic functional equation of Apollonius type, and quadratic. We also prove the Hyers–Ulam stability of quadratic mappings of n–Apollonius type in Banach spaces.

2. Main results

Theorem 2.1. *(n–Apollonius' identity) Let X be an inner product space with norm $\|.\|$ introduced by its inner product $\langle .,.\rangle$. For a fixed positive integer $n \geq 2$, we have*

$$\sum_{i=1}^{n}\|z-x_i\|^2=\frac{1}{n}\sum_{\substack{1\leq i,j\leq n\\ j<i}}\|x_i-x_j\|^2+n\Big\|z-\frac{1}{n}\sum_{i=1}^{n}x_i\Big\|^2 \tag{2.1}$$

for all $z, x_1, \ldots, x_n \in X$.

Theorem 2.2. *A mapping $Q : X \to Y$ is a quadratic mapping of n–Apollonius type ($n \geq 2$), i.e., Q satisfies*

$$\sum_{i=1}^{n} Q(z-x_i)=\frac{1}{n}\sum_{\substack{1\leq i,j\leq n\\ j<i}} Q(x_i-x_j)+nQ\Big(z-\frac{1}{n}\sum_{i=1}^{n}x_i\Big) \tag{2.2}$$

for all $z, x_1, \ldots, x_n \in X$, if and only if Q is a quadratic mapping of Apollonius type and quadratic mapping.

2.1. **Hyers–Ulam stability of a quadratic mapping of n–Apollonius type.** Throughout this section, let X be a normed space with norm $\|.\|_X$ and Y a Banach space with norm $\|.\|_Y$.

For a fixed integer $n \geq 2$ and given a mapping $Q : X \to Y$, we define $D_nQ : X^{n+1} \to Y$ by

$$D_nQ(x_1, x_2, \ldots, x_n, z) := \sum_{i=1}^{n} Q(z - x_i) - \frac{1}{n} \sum_{\substack{1\leq i,j\leq n \\ j<i}} Q(x_i - x_j) - nQ\Big(z - \frac{1}{n}\sum_{i=1}^{n} x_i\Big).$$

Theorem 2.3. *Let l and m be integers with $1 \leq l < m$ and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is a mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} (\frac{l}{m})^{2i} \varphi\Big(\underbrace{0, \ldots, 0}_{l-times}, (\frac{m}{l})^i z, \ldots, (\frac{m}{l})^i z\Big) < \infty, \tag{2.3}$$

$$\lim_{i\to\infty} (\frac{l}{m})^{2i} \varphi\Big((\frac{m}{l})^i x_1, \ldots, (\frac{m}{l})^i x_m, (\frac{m}{l})^i z\Big) = 0 \tag{2.4}$$

and

$$\Big\|D_mQ(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \delta + \varphi(x_1, \ldots, x_m, z) \tag{2.5}$$

for all $z, x_1, \ldots, x_m \in X$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|Q(x) - T(x) - \frac{2l^2 - m^2 - m}{2(m^2 - l^2)} Q(0)\Big\|_Y \leq \frac{m\delta}{m^2 - l^2} + \frac{1}{m}\widetilde{\varphi}\Big(\frac{m}{l}x\Big) \tag{2.6}$$

for all $x \in X$.

Theorem 2.4. *Let l and m be integers with $1 \leq l < m$ and suppose that $Q : X \to Y$ is a mapping for which there exists a function $\varphi : X^{m+1} \to [0, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} (\frac{m}{l})^{2i} \varphi\Big(\underbrace{0, \ldots, 0}_{l-times}, (\frac{l}{m})^i z, \ldots, (\frac{l}{m})^i z\Big) < \infty, \tag{2.7}$$

$$\lim_{i\to\infty} (\frac{m}{l})^{2i} \varphi\Big((\frac{l}{m})^i x_1, \ldots, (\frac{l}{m})^i x_m, (\frac{l}{m})^i z\Big) = 0 \tag{2.8}$$

and

$$\Big\|D_mQ(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \varphi(x_1, \ldots, x_m, z) \tag{2.9}$$

for all $z, x_1, \ldots, x_m \in X$. Let $Q(0) = 0$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\big\|Q(x) - T(x)\big\|_Y \leq \frac{m}{l^2}\widetilde{\varphi}(x) \tag{2.10}$$

for all $x \in X$.

Theorem 2.5. *Let $m \geq 3$ be an odd integer and let δ be a nonnegative real number. Suppose that $Q : X \to Y$ is an even mapping for which there exists a function $\varphi : X^{m+1} \to [-\delta, \infty)$ such that*

$$\widetilde{\varphi}(z) := \sum_{i=0}^{\infty} \alpha^i \varphi\Big(\gamma^i z, 0, \gamma^i z, 0, \ldots, 0, \gamma^i z, \gamma^i z\Big) < \infty, \tag{2.11}$$

$$\lim_{i\to\infty} \alpha^i \varphi\Big(\gamma^i x_1, \gamma^i x_2, \ldots, \gamma^i x_m, \gamma^i z\Big) = 0 \tag{2.12}$$

and

$$\Big\|D_m Q(x_1, x_2, \ldots, x_m, z)\Big\|_Y \leq \delta + \varphi(x_1, \ldots, x_m, z) \tag{2.13}$$

for all $x_1, \ldots, x_m, z \in X$, where $\alpha = \frac{(m-1)^2}{4m^2}$ and $\gamma = \frac{2m}{m-1}$. Then there exists a unique quadratic mapping of m–Apollonius type $T : X \to Y$ such that

$$\Big\|T(x) - \alpha Q(x) + \frac{\alpha\beta}{1-\alpha} Q(0)\Big\|_Y \leq \frac{\alpha}{m(1-\alpha)}\delta + \frac{\alpha}{m}\widetilde{\varphi}(\gamma x) \tag{2.14}$$

for all $x \in X$, where $\beta = \frac{m^2+4m-1}{4m^2}$.

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RELATIVE DISTANCE EIGENVALUE FOR SPECIAL MATRIX

ALI MOHAMMAD NAZARI[1*] AND SOMAYEH ZEYNALI[2]

Abstract. In this paper we find upper bound for relative distance eigenvalue $A+E$, where E is a perturbation matrix and A may be a special matrix, positive definite, singular, normal and Hermitian.

1. Introduction and preliminaries

Theorem of Bauer-Fike [1] gives the following bound for $\min|\lambda_i - \mu|$, where A is diagonalizable matrix with eigendecomposition $A = X\Lambda X^{-1}$, λ_i are the eigenvalue of A and μ is an eigenvalue of $A+E$ then

$$\min|\lambda_i - \mu| < \kappa_2(X)\|E\|_2,$$

where $\kappa_2 = \|X\|_2\|X^{-1}\|_2$ is the condition number of eigenvector matrix X. If A is nonsingular, this leads to relative distance:

$$\min \frac{|\lambda_i - \mu|}{|\lambda_i|} < \kappa_2(X)\|A^{-1}E\|.$$

If A be a singular matrix form [2] we have the following theorem for relative perturbation eigenvalue.

Theorem 1.1. *If $\mu > \kappa_2(X)\|E\|_2$ (i.e. μ is too large in magnitude for zero eigenvalue of A to satisfy Bauer-Fike bound), then*

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} < \kappa_2(X)\|A^{-1}E\|_2\sqrt{1+\alpha^2},$$

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Key words and phrases. Baur-Fike Theorm, perturbation of matrix, relative distance of matrix.

* Speaker.

where

$$\alpha = \frac{\kappa_2(X)\|E\|_2}{\sqrt{|\mu|^2 - (\kappa_2(X)\|E\|_2)^2}}$$

and A^+denote the pseudo-inverse of A:

In this paper we assume A be a normal or Hermition or positive definite and get a simpler bound for relative distance eigenvalue.

2. Relative distance for normal matrix

Theorem 2.1. *Let A be an $n \times n$ normal matrix and μ is an eigenvalue of $A + E$ such that $\mu > \|E\|_2$, then*

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} < \frac{|\mu| \|A^{-1}E\|_2}{\sqrt{\mu^2 - (\|E\|_2)^2}}.$$

Proof. . Since A is a normal matrix, then by Schur decomposition theorem there exits a unitary matrix X such that

$$A = X \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} X^*$$

where Λ_1 is an upper triangular matrix with eigenvalues of A in its main diagonal. By proof of theorem 1 of [2] we can write the proof of this theorem and instead of X^{-1}, we can write X^* with $\|X\|_2 = \|X^*\|_2 = 1$, since X is orthogonal matrix . Then $\kappa_2(X) = 1$ and consequently

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} < \frac{|\mu| \|A^{-1}E\|_2}{\sqrt{\mu^2 - (\|E\|_2)^2}}.$$

□

3. Relative distance for non-diagonalizable singular matrix.

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Theorem 3.1. *Let A be a singular and nondiagonalizable matrix. Let $A = Q^*UQ$ be the Schur decomposition of A and $U = D + N$, λ_i be an eigenvalue of A. Let μ be an eigenvalue of $A + E$ and u be the corresponding eigenvector of μ. If $|\mu| \geq \|E\| + \|N\|$,then*

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} \leq [\|A^+\| + |\mu| \, \|U^{-1}\|]\sqrt{1 + \alpha^2} + \mu\lambda_{(min)}(A),$$

where

$$\alpha = \frac{\|E\|_2}{\sqrt{|\mu^2| - (\|E\|_2)^2}}.$$

Proof. Assume without loss of generality that

$$A = Q \begin{pmatrix} \Lambda_1 & 0 \\ 0 & 0 \end{pmatrix} Q^*,$$

where Λ_1 is an $(m \times m)$ nonsingular matrix. Let u be an eigenvector corresponding to μ and define

$$\begin{pmatrix} E_{11} & E_{12} \\ E_{11} & E_{11} \end{pmatrix} = Q^*EQ = T \qquad \text{and} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = Q^*u,$$

where E_{11} is an $m \times m$ matrix and u_1 is an m-vector. Since $(A+E)u = \mu u_1$,

$$\mu \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \mu Q^* u = Q^* \mu u = Q^*(A+E)u = Q^*(A+E)QQ^*u =$$

$$[Q^*AQ + Q^*EQ]Q^*u = [U+T]Q^*u[\mu(u_1,u_2) = [U+T]Q^*u] \Rightarrow$$

$$\mu u_2 = [U+T]Q^*u = \left[\begin{pmatrix} E_{11} & E_{12} \\ E_{11} & E_{11} \end{pmatrix} + \begin{pmatrix} \Lambda & N-1 \\ 0 & N_2 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \right],$$

where

$$\Lambda = \operatorname{diag}(\lambda_1, ..., \lambda_2).$$

Taking norm of each side,

$$|\mu| \|E\|_2 \leq (\|E\|_2 + \|N\|_2)\sqrt{\|u_1\|^2 + \|u_2\|^2}.$$

Hence

$$\mu^2 - ((\|u_2\|)_2)^2 - ((\|u_2\|)_2)^2(\|E\|_2 + \|N\|_2)^2 \leq (\|E\|_2 + \|N\|_2)^2((\|u_1\|)_2)^2$$

$$((\|u_2\|)_2)^2 \leq \left[\frac{(\|E\|_2 + \|N\|_2)^2}{(\mu^2 - (\|E\|_2 + \|N\|_2)^2)} \right]$$

$$((\|u_2\|)_2)^2 \leq (\alpha^2)((\|u_1\|)_2)^2$$

$$Q^*(A^+(A-\mu I))Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - QA^+EuQ^*(I-\mu A^+)Q \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$= Q^*(QQ^* - \mu QU^+Q^*)QQ^*u = (I - \mu U^+) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$$

$$(I - \mu U^+)) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = -QA^+Eu$$

$$\|(I - \mu U^+) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \| \leq \|(I - \mu U^+)|\mu| \|U^+ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \| + \||\mu| D^+ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \|$$

$$\|(I - \mu U^+) \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \| \leq \|A^+E\| \|u\| |\mu| U^+ \| |\mu| D^+ \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \|$$

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} \|u_1\| \leq (\|A^+E\| + |\mu| \|U^+\|) \| \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \| + |\mu| D^+ \|u_1\|$$

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} \|u_1\| \leq (\|A^+E\| + |\mu| \|U^+\|)\sqrt{1+\alpha^2} \|u_1\| + \lambda_{min} \|u_1\|$$

$$\min_{\lambda_i \neq 0} \frac{|\lambda_i - \mu|}{|\lambda_i|} \leq (\|A^+E\| + |\mu| \|U^+\|)\sqrt{1+\alpha^2} \|u_1\| + \lambda_{min}.$$

□

Theorem 3.2. *Let* $Q^*AQ = D + N$ *be the Schur decomposition of* $A \in A^{n \times n}$. *If* $\mu \in \lambda(A+E)$ *and* p *is the smallest positive number such that* $|N|^p = 0$ *then* $\min_{\lambda \in \lambda(A)} \leq |\lambda - \mu| \leq \max |\vartheta, -\vartheta^{\frac{1}{p}}|$, *where* $\vartheta = \|E\|_2 \Sigma(\|N\|_2)^k$.

3. □

Theorem 3.3. *Let $A \in A^{n\times n}$ and nonsingular and λ_i, $i = 1...n$ be eigenvalues of A, μ be an eigenvalue of $A + E$, let $Q^*AQ = D + N$ be the the Schur decomposition of A, and p be the smallest positive number such that $|N|^p = 0$. Then*

$$\min \frac{|\lambda_i - \mu|}{\lambda_i} \leq \max |\vartheta - \vartheta^{\frac{1}{p}}|,$$

where

$$\vartheta = \|E\|_2 \Sigma(\lambda_{max})^{k+1}(\|N\|)^k.$$

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AN OPTIMIZATION METHOD FOR SOLVING THE TWO-DIMENSIONAL NONLINEAR FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND

ALI REZA NAZEMI[1]* AND MOHAMMAD HADI FARAHI [2]

Abstract. In this numerical approach, we obtain the solution of the Two-Dimensional Nonlinear Fredholm Integral Equations of the Second Kind problem by converting the problem to an optimal moment problem. The moment problem is modified into one consisting of the minimization of a positive linear functional over a set of Radon measures. Then we obtain an optimal measure which is approximated by a finite combination of atomic measures and by using atomic measures we change this one to a semi-infinite dimensional linear programming problem. We approximate the latter one by a finite dimensional linear programming problem.

1. Introduction and preliminaries

In this numerical approach, we obtain the solution of the original problem by converting the problem to an optimal moment problem. The moment problem is modified into one consisting of the minimization of a positive linear functional over a set of Radon measures. Then we obtain an optimal measure which is approximated by a finite combination of atomic measures, and by using atomic measures we change this one to a semi-infinite dimensional linear programming problem. We approximate the latter one by a finite dimensional linear programming problem.

2. Main results

There has been much work on developing and analyzing numerical methods for solving Fredholm integral equations of the second kind ([2]). Consider the following

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problem

$$u(x,y) = f(x,y) + \int_a^b \int_c^d k(x,y,s,t,u(s,t))dtds,\ (x,y) \in D \tag{2.1}$$

where $u(x,y)$ is unknown function, $f(x,y)$ and $k(x,y,s,t,u)$ are given continuous functions defined, respectively, on $[a,b] \times [c,d]$ and $E = \{D \times D \times (-\infty,\infty)\}$ with $k(x,y,s,t,u)$ nonlinear in u.
Let the interval $I = [a,b]$ and $J = [c,d]$ is divided to M and N subintervals, $I_i = [x_{i-1}, x_i], J_j = [y_{j-1}, y_j]$ respectively, where $[a,b] = \bigcup_{i=1}^{M} I_i$ and $[c,d] = \bigcup_{j=1}^{N} J_j$. Then, we convert the integral equation (1) on $I \times J$ to the following system :

$$\int_a^b \int_c^d k(x_i,y_j,s,t,u(s,t))dtds - u(x_i,y_j) = -f(x_i,y_j), \tag{2.2}$$

for $i = 1,2,\dots,M$ and $j = 1,2,\dots,N$. Now, we define the following problem:

$$\text{minimize} \quad \int_a^b \int_c^d g(s,t,u(s,t))\ dtds \tag{2.3}$$

subject to

$$\int_a^b \int_c^d k(x_i,y_j,s,t,u(s,t))dtds - u(x_i,y_j) = -f(x_i,y_j), \tag{2.4}$$

for $i = 1,2,\dots,M$ and $j = 1,2,\dots,N$, where the artificial function $g(s,t,u(s,t))$ is an arbitrary continuous function. So, finding a solution for the system (2.1) is equivalent as finding a solution of the optimization problem (2.3)-(2.4).

Definition 2.1. The trajectory function $u(\cdot,\cdot) : [a,b] \times [c,d] \to U \subset I\!R$ called admissible if is absolutely continuous and the constrains of the problem (3)-(4) are satisfied.

Now, define $a_{ij} = u(x_i,y_j) - f(x_i,y_j)$. Hence, our optimization problem (2.3)-(2.4) is reduced to the following problem

$$\text{minimize} \quad \int_a^b \int_c^d g(s,t,u(s,t))\ dtds \tag{2.5}$$

subject to

$$\int_a^b \int_c^d k(x_i,y_j,s,t,u(s,t))dtds = a_{ij},\ (i = 1,2,\cdots,M)\ (j = 1,2,\cdots,N). \tag{2.6}$$

Let $\Omega = I \times J \times U$ and $C(\Omega)$ be the space of all continuous functions on Ω. Now we replace the moment minimization problem (2.5)-(2.6) to another one.

Theorem 2.2. *The mapping* $\Lambda : h \longrightarrow \int_a^b \int_c^d h(s,t,u(s,t))dtds,\ h \in C(\Omega)$, *defines a positive, bounded, linear functional on* $C(\Omega)$.

Theorem 2.3. *(the Riesz representation theorem ([4])) There exists a unique positive Radon measure* μ *on* Ω *such that*

$$\Lambda(h) = \int_\Omega h d\mu \equiv \mu(h),\ \forall\ h \in C(\Omega). \tag{2.7}$$

These measures μ are required to have certain properties. First by (2.7) $|\mu(h)| \leq ST \sup_\Omega |h(s,t,u(s,t))|$ where $T = b - a$ and $S = d - c$. Hence $\mu(1) \leq ST$. Also, by (6)-(7) we have $\mu(k(x_i, y_j, s, t, u(s,t))) = a_{ij}$. Finally, we consider functions $\theta \in C(\Omega)$ which do not depend on u, that is $\theta(s,t,u_1) = \theta(s,t,u_2)$ for all $s \in [a,b]$, $t \in [c,d]$, $u_1, u_2 \in U$, where $u_1(\cdot,\cdot) \neq u_2(\cdot,\cdot)$. Then, $\int_\Omega \theta d\mu = \int_a^b \int_c^d \theta(s,t,u(s,t))dtds = \alpha_\theta$, where u is an arbitrary element of U.

Let $M^+(\Omega)$ be the set of all positive Radon measures on Ω. The set Q is defined as a subset of $M^+(\Omega)$ such that $Q = S_1 \cap S_2 \cap S_3$ where
$S_1 = \{\mu \in M^+(\Omega) : \mu(1) \leq ST\}$
$S_2 = \{\mu \in M^+(\Omega) : \mu(k(s,t,x_i,y_j,u(s,t))) = a_{ij}, \}$
$S_3 = \{\mu \in M^+(\Omega) : \mu(\theta) = \alpha_\theta, \theta \in C(\Omega)$ and independent of $u.\}$.

Theorem 2.4. *Suppose we topologize the space $M^+(\Omega)$ by the weak*-topology. Thus, Q is compact convex set.([4])*

Now the original minimization problem is replaced by one in which we seek the minimum of

$$I(\mu) = \int_\Omega g d\mu \equiv \mu(g) \tag{2.8}$$

over the compact, convex, set Q.

Theorem 2.5. *The measure-theoretic problem, which consists in finding the minimum of the functional (2.8) over the set Q of $M^+(\Omega)$, attains its minimum, say $\mu*$, in the set Q. ([3])*

Lemma 2.6. *i)The set Q is nonempty.*

ii) The solution of the original optimal moment problem (2.5)−(2.6) is equivalently the same as the solution of (2.8) over Q and visa-versa.

We obtain an approximation to the optimal measure $\mu^* \in M^+(\Omega)$ that is defined in Theorem (1.5). We consider the following problem

$$\text{minimize} \qquad I(\mu) = \mu(g) \tag{2.9}$$

subject to:

$$\begin{cases} \mu(1) \leq ST, \\ \mu(k(s,t,x_i,y_j,u(s,t))) = a_{ij}, \ (i = 1,2,\cdots,M) \ (j = 1,2,\cdots,N), \\ \mu(\theta) = \alpha_\theta, \ \theta \in C(\Omega) \text{ and independent of } u\}. \end{cases} \tag{2.10}$$

In the above linear programming problem, we choose the only a finite number GH of functions θ. Now, we divide I to m, J to n and U to p subintervals. In this way, we obtain a grid for Ω and rename every subrectangle obtained as Ω_k. We take a member from each subrectangle of Ω_k and denote it by $z_k = (s_k, t_k, u_k)$. As a result in ([**?**]), is shown that the optimal measure μ^* that satisfy the constraints of (2.10), and minimize the functional (2.9) has the following form $\mu^* = \sum_{k=1}^R \beta_k^* \delta(z_k^*)$ where $R = mnp$ is the number of points in the above partition, $z_k^* \in \Omega$, the coefficients $\beta_k^* \geq 0$, $k = 1,2,\cdots,R$, are unknown and $\delta(z_k^*)$ for each k, denote a unitary atomic measure with support the singleton set $\{z_k^*\}$, such that $\delta(z_k^*)(F) = F(z_k^*)$ for all $F \in C(\Omega)$. Note that we choose $\theta_{st}(s,t,u)$ as follows

$$\theta_{st}(s,t,u) = \begin{cases} 1 & \text{if } t \in J_{st} \\ 0 & \text{otherwise} \end{cases}$$

where $J_{st} = [\frac{(s-1)T}{G}, \frac{sT}{G}] \times [\frac{(t-1)T}{H}, \frac{tT}{H}]$, $(s = 1, 2, \ldots, G)$, $(t = 1, 2, \ldots, H)$. So $\alpha_{\theta_{st}} = \int_{J_{st}} \theta_{st}(s, t, u(s,t))dt = \frac{ST}{GH}$. Consequently the problem (9)-(10) convert to the problem minimizing $\sum_{k=1}^{R} g(z_k)\beta_k$ subject to

(2.11)

$$\begin{cases} \sum_{k=1}^{R} \beta_k k_{ij}(z_k) - u(x_i, y_j) = -f(x_i, y_j), \ (i = 1, 2, \cdots, M) \ (j = 1, 2, \ldots, N), \\ \sum_{k=1}^{R} \beta_k = ST, \\ \beta_k \geq 0, (k = 1, 2, \ldots, R), \ u(x_i, y_j) \text{ is free}, (i = 1, 2, \ldots, M)(j = 1, 2, \ldots, N). \end{cases}$$

By using of slack variables $u(x_i, y_j)$ that has been obtained by solving the above linear programming, we can get approximate solution for the original problem (2.1) by interpolating a surface on support point $(x_i, y_j, u(x_i, y_j))$.

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APPLICATIONS OF MAXIMALITY IN THE DECOMPOSITION OF HILBERT SPACES

ALI PARSIAN

ABSTRACT. The Closed subsets of Hilbert spaces have usually a basic role in the theory of these topological vector spaces. In this talk we introduce a kind of closed subsets $D \subseteq H \oplus H$ for any Hilbert space H, which has also a kind of maximality and for which every $h \in H$ has a unique representation $h = x - y$ for some $(x, y) \in D$. As a consequence, we prove the standard representation of the Hilbert space H as the direct sum of the closure of M and its orthogonal complement for any subspace $M \subseteq H$. We also discuss some applications.

1. INTRODUCTION AND PRELIMINARIES

The decomposition theorem is a very useful uniqueness theorem in the Hilbert space theory. In this paper we study some basic properties of some subspaces of a complex Hilbert space H, which have a kind of maximality with respect to a complex form. This leads to some decomposition of elements in Hilbert space with respect to these maximal subspaces. As a useful application, it is shown that any $h \in H$ has a unique representation $h = x - Sx$ for any skew Hermitian linear map and some $x \in H$. Then we generalized this theorem to all densely defined skew Hermitian linear maps.

2. GENERALIZED MAXIMAL ISOTROPIC SUBSPACE

Let $(H, <, >)$be a complex Hilbert space. By a bilinear form on $H \oplus H$, we mean a continuous bilinear mapping $g : H \oplus H \longrightarrow C$. A maximal isotropic subspace with respect to g,is a subspace $D \subseteq H \oplus H$ such that

$$g((x, y), (x^{'}, y^{'})) = 0, \forall (x, y), (x^{'}, y^{'}) \in D$$

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Furthermore D has no proper extension with the above property.

Theorem 2.1. *For any Hilbert space H, the set Ω consists of all maximal isotropic subspaces $D \subseteq H \oplus H$ is nonempty and all of its elements are complete.*

Proof. The set I of all isotropic subspaces of $H \oplus H$ is nonempty and partially ordered by inclusion relation in which every chain has an upper bound,so I possesses a maximal element by Zorn's lemma and I is nonempty. Now let $D \in I$, $(u, v) \in \overline{D}$and (x_n, y_n) be a sequence of elements of D tends to (u, v) with respect to $H \oplus H$ norm, then for any $(x, y) \in D$ we have,

$$g((x, y), (u, v)) = g((x, y), lim(x_n, y_n)) = limg((u, v), (x_n, y_n)) = 0$$

Now maximality of D implies that $(u, v) \in D$. *i.e.,* is a closed (and hence complete) subspace of $H \oplus H$. ☐

Now let g be a bilinear form on $H \oplus H$defined by $g((x, y), (x', y')) =< x, y' > + < y, x' >$,and D be a maximal isotropic subspace defined by g and $p_i (i = 1, 2)$ be the first and second projections, then we have,

Theorem 2.2. *With the above notions, the linear map $p_1 - p_2 : D \longrightarrow H$ is an isometry.*

Proof. Let $z \perp (p_1 - p_2)(D)$, by decomposition theorem on Hilbert spaces, there are $(x, y), (x', y') \in D$ such that $(z, -z) = (x, y) + (y', x')$. Therefore,

$$2 \parallel z \parallel^2 =< (z, -z), (z, -z) >=< z, x - y > + < z, y' - x' >= 0$$

i.e.,z=0, hence $(p_1 - p_2)(D)$ is dense in $H.p_1 - p_2$ is also norm preserving, because for $(x, y) \in D$ we have,

$$\parallel (x, y) \parallel^2 =< (x, y), (x, y) >=$$
$$=< (x, x), (y, y) >=\parallel x \parallel^2 + \parallel y \parallel^2 =< x - y, x - y >=\parallel x - y \parallel^2$$

consequently if $(p_1 - p_2)(x_n)$ be a Cauchy sequence in H, then (x_n) is also a Cauchy sequence in D. Hence $(p_1 - p_2)(D) = H$. ☐

Theorem 2.3. *The map $p_1 - p_2 : D \longrightarrow H$ is an isomorphism of Hilbert spaces.*

Proof. By Theorem 2.2 and Banach's theorem it is enough to show that the map is injective. Let $z = (x, y) \in ker(p_1 - p_2)$, then $x - y = 0$ and $(x, x) \in D$, therefore $2 \parallel x \parallel^2 =< (x, x), (x, x) >= 0$; i.e. $z = 0$. ☐

Theorem 2.4. *Let $D \subseteq H \oplus H$ be a maximal isotropic subspace, then every $h \in H$ has a unique representation $h = x - y$ for some $(x, y) \in D$. Moreover, $\parallel h \parallel^2 =\parallel x \parallel^2 + \parallel y \parallel^2$.*

Proof. By Theorem 2.2, $p_1 - p_2$ is an isometry. Hence $h = x - y$ for some $(x, y) \in D$ with required identity on norms. ☐

Now the standard decomposition theorem is a straightforward consequence,

Theorem 2.5. *Let M be a closed subspace of The complex Hilbert space H. Then every $h \in H$ has a unique representation $h = x + y$ for some $x \in M$ and $y \in M^{\perp}$.*

Proof. Clearly $D = M \times M^{\perp}$ is an isotropic subspace of $H \oplus H$. Assume that $g((u, v)$ $, (x, y)) = 0$ for all $x \in M$ and $y \in M^{\perp}$. Consequently $u \in \overline{M} = M$ and $v \in M^{\perp}$. So D is a maximally isotropic subspace of $H \oplus H$. Now Theorem 2.4 implies that $h = x - t$ for some $x \in M$ and $t \in M^{\perp}$; i.e. $h = x + y$ for $x \in M$ and $y = -t \in M^{\perp}$. □

Theorem 2.6. *Let $S : H \longrightarrow H$ be a skew Hermitian linear map. Then any $h \in H$ has a unique representation $h = x - Sx$ for some $x \in H$ such that $\| h \|^2 = \| x \|^2 + \| Sx \|^2$.*

Proof. $D = graph(S)$ is a maximal isotropic subspace of $H \oplus H$ [1]. Now this is an immediate consequence of Theorem 2.4. □

Theorem 2.7. *Let $S : H \longrightarrow H$ be a densely defined skew Hermitian linear map. Then $graph(S)$ is closed.*

Proof. Let (x_n, y_n) be a sequence in $graph(S)$ tends to (x, y) and $u \in Dom(S)$, then $< u, -Sx_n > = < Su, x_n >$ therefore,

$$< u, -y > = < Su, x >, x \in Dom(S) = Dom(S^*)$$

Thus $< u, -y > = < u, -Sx >$ and $u \perp y - Sx$ for all $u \in Dom(S)$. Since $Dom(S)$ is dense in H, we have $y - Sx = 0$ and $(x, y) \in graph(S)$. □

Theorem 2.8. *Let $S : H \longrightarrow H$ be a densely defined skew Hermitian linear map. Then $graph(S)$ is a maximal isotropic subspace of $H \oplus H$.*

Proof. By Theorem 2.7, $graph(S)$ is a closed subspace of $H \oplus H$. Let $\{(a, b)\} \cup graph(S)$ be an isotropic subset of $H \oplus H$. Then,

$$g((a, b), (x, Sx)) = < a, Sx > + < b, x > = 0$$

for all $x \in Dom(S)$. Furthermore without loss of generality we can assume,$< a, x > + < b, Sx > = 0$ consequently,

$$< a + b, x + Sx > = < a, x > + < b, Sx > + < a, Sx > + < b, x > = 0$$

for all $x \in Dom(S)$. Let $u = a + b$, then $0 = < x + Sx, u > = < x, u > + < Sx, u >$ and $< x, -u > = < Sx, u >$, therefore $u \in Dom(S^*) = Dom(S)$ and $< x, -u > = < x, -Su >$ for all $x \in Dom(S)$. Consequently,$Su = u$ and,

$$\| u \|^2 = < u, u > = < Su, u > = < u, -Su > = - \| u \|^2$$

or $u = 0$ and $a = -b$. Since $graph(S) \cup \{(a, b)\}$ is isotropic subset, $a = b = 0$, and $graph(S)$ is maximally isotropic subspace of $H \oplus H$. □

Theorem 2.9. *Let $S : H \longrightarrow H$ be a densely defined skew Hermitian linear map. Then every $h \in H$ has a unique representation $h = x - Sx$,for some $x \in H$ such that,*

$$\| h \|^2 = \| x \|^2 + \| Sx \|^2$$

Proof. $D = graph(S)$ is a maximal isotropic subspace of $H \oplus H$ by Theorem 2.8. Thus $p_1 - p_2 : D \longrightarrow H$is an isometry by Theorem 2.2. Hence $h = x - Sx$ for some $x \in H$ and the above identity on norms. □

2.1. **Application.** Let $H = L^2(R)$ and $S = \frac{d}{dx}$ be the differential operator defined on the space $H = C_0^1(R)$, the space of all continuously differentiable real functions with compact support. Using integration by parts,it can be shown that S is a skew Hermitian linear map. Thus by Theorem 2.10 any $h \in L^2(R)$ has a unique representation $h = f - \frac{df}{dx}$ for some $f \in C_0^1(R)$.

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D-BOUNDED SETS IN GENERALIZED PROBABILISTIC 2-NORMED SPACES

ALIREZA POURMOSLEMI[1]* AND MEHDI SALIMI[2]

Abstract. K. Menger introduced the notion of probabilistic metric spaces. The idea of Menger was to use distribution function instead of nonnegative real numbers as values of the metric. The concept of probabilistic normed spaces were introduced by Šerstnev. Then I. Goleţ defined generalized probabilistic 2-normed space (briefly GP-2-N space). In this paper, we defined bounded sets in GP-2-N space and we studied the relationship among D-bounded sets and proved some result in this space.

1. Introduction and preliminaries

New definition of probabilistic normed spaces was studied by Alsina, Schweizer and Sklar [1]. D-bounded sets in probabilistic normed spaces were studied by B. Lafuerza-Guillén [3]. A distribution function (briefly, a d.f.) is a function F from the extended real line $\overline{\mathbb{R}} = [-\infty, +\infty]$ into the unit interval $I = [0, 1]$ that is nondecreasing and satisfies $F(-\infty) = 0,\ F(+\infty) = 1$. The set of all d.f.'s will be denoted by Δ and the subset of those d.f.'s such that $F(0) = 0$, will be denoted by Δ^+. $D^+ \subseteq \Delta^+$ is defined as follows:

$$D^+ = \{F \in \Delta^+ : l^- F(+\infty) = 1\},$$

where $l^- f(x)$ denotes the left limit of the function f at the point x. by setting $F \leq G$ whenever $F(x) \leq G(x)$ for all x in $\mathbb{R}$, the maximal element for Δ^+ in this order is the d.f. given by

$$\varepsilon_0(x) = \begin{cases} 0, & \text{if } x \leq 0, \\ 1, & \text{if } x > 0. \end{cases}$$

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* Speaker.

A *triangle function* is a binary operation on Δ^+, namely a function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ that is associative, commutative, nondecreasing and which has ε_0 as unit.

Definition 1.1. Let L, M be two real linear spaces of dimension greater than one, and let $\mathcal{F}$ be a function defined on the Cartesian product $L \times M$ into Δ^+ satisfying the following properties:

(1) $F_{\alpha x,y}(t) = F_{x,\alpha y}(t) = F_{x,y}(\frac{t}{|\alpha|})$, for every $t > 0, \alpha \in \mathbb{R} - \{0\}$ and $(x, y) \in L \times M$.
(2) $F_{x+y,z} \geq \tau(F_{x,z}, F_{y,z})$, for every $x, y \in L$ and $z \in M$.
(3) $F_{x,y+z} \geq \tau(F_{x,y}, F_{x,z})$, for every $x \in L$ and $y, z \in M$.

The function $\mathcal{F}$ is called a *generalized probabilistic 2-norm* on $L \times M$ and the triple $(L \times M, \mathcal{F}, \tau)$ is called a *generalized probabilistic 2-norm space(briefly GP-2-N space).*

2. Main results

Definition 2.1. Let a nonempty set $A \times B$ be in a GP-2-N space $(L \times M, \mathcal{F}, \tau)$, then its probabilistic radius $R_{A\times B}$ is define by

$$R_{A\times B}(x) = \begin{cases} l^-\varphi_{A\times B}(x), & \text{if } x \in [0, +\infty), \\ 1, & \text{if } x = +\infty, \end{cases}$$

where

$$\varphi_{A\times B}(x) := inf\{F_{p,q}(x) : p \in A, q \in B\}.$$

Definition 2.2. A nonempty set $A \times B$ in a GP-2-N space $(L \times M, \mathcal{F}, \tau)$ is said to be:
(a) *Certainly bounded*, if $R_{A\times B}(x_0) = 1$ for some $x_0 \in (0, +\infty)$,
(b) *Perhaps bounded* if one has $R_{A\times B}(x) < 1$ for every $x \in (0, +\infty)$, and $l^-R_{A\times B}(+\infty) = 1$,
(c) *Perhaps unbounded*, if $R_{A\times B}(x_0) > 0$ for some $x_0 \in (0, +\infty)$, and $l^-R_{A\times B}(+\infty) \in (0, 1)$,
(d) *Certainly unbounded*, if $l^-R_{A\times B}(+\infty) = 0$.
Moreover, A is said to be *D-bounded* if either(a) or (b) holds.

Lemma 2.3. *Let $(L \times M, \mathcal{F}, \tau)$ be a GP-2-N space and $A \times B \subseteq L \times M$. Then $A \times B$ is a D-bounded set if,and only if,*

$$\lim_{x\to+\infty} \varphi_{(A\times B)}(x) = 1.$$

□

Theorem 2.4. *A set $A \times B$ in the GP-2-N space $(L \times M, \mathcal{F}, \tau)$ is D-bounded if,and only if, there exists a d.f. $G \in D^+$ such that $F_{a,b} \geq G$ for every $a \in A, b \in B$.*

□

Proposition 2.5. *Let $(L \times M, \mathcal{F}, \tau)$ be a GP-2-N space. If $|\alpha| \leq |\beta|$, then $F_{\beta p,q} \leq F_{\alpha p,q}$ for every $(p, q) \in L \times M$ and $\alpha, \beta \in \mathbb{R} - \{0\}$.*

□

If $A \times B$ is a D-bounded set then $\alpha A \times B$ need not be D-bounded set, but this will hold under suitable conditions, as is shown in the next theorem.

Theorem 2.6. *Let $(L \times M, \mathcal{F}, \tau)$ be a GP-2-N space and $A \times B$ be a D-bounded subset in $L \times M$. The set $\alpha A \times B := \{(\alpha p, q) : p \in A, q \in B\}$ is also D-bounded for every fixed $\alpha \in \mathbb{R} - \{0\}$ if D^+ is a closed set under τ, i.e. $\tau(D^+ \times D^+) \subseteq D^+$.*

□

Theorem 2.7. *Let $(L \times M, \mathcal{F}, \tau)$ be a GP-2-N space. Suppose $A \times B$ and $C \times B$ be two nonempty and D-bounded sets in $L \times M$.Then $(A + C) \times B$ is a D-bounded set if D^+ is a closed set under τ, i.e. $\tau(D^+ \times D^+) \subseteq D^+$.*

Proof. For every $(a, b) \in A \times B$ and $(c, b) \in C \times B$, we have $(a + c, b) \in (A + C) \times B$. Therefore

$$F_{a+c,b} \geq \tau(F_{a,b}, F_{c,b}) \geq \tau(F_{a,b}, R_{C\times B}) \geq \tau(R_{A\times B}, R_{C\times B}).$$

This implies that

$$R_{(A+C)\times B} \geq \tau(R_{A\times B}, R_{C\times B}).$$

Now by the fact that $\tau(D^+ \times D^+) \subseteq D^+$ we have $\tau(R_{A\times B}, R_{C\times B})$ is in D^+. It means $l^- R_{(A+C)\times B}(+\infty) = 1$. This completes the proof. □

Corollary 2.8. *If $A \times B$ and $C \times B$ are two nonempty and D-bounded sets in $L \times M$ and $\tau(D^+ \times D^+) \subseteq D^+$, then $\alpha(A + C) \times B$ is a D-bounded set for every $\alpha \in \mathbb{R} - \{0\}$.*

□

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WAVE PACKET SYSTEMS

ASGHAR RAHIMI

ABSTRACT. In this paper, we present a sufficient condition for a generalized shift-invariant system to be a Bessel sequence or even a frame for $L^2(\mathbb{R}^d)$. In particular, this leads us to a sufficient condition for a wave packet system, systems in the form $\{D_{A_j} T_{Fk} E_{c_m} g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$, to form a frame. On the other hand, we show that certain natural conditions on the parameters of such a system exclude the frame property. We also characterize the dual pairs of these systems.

1. Introduction and Preliminaries

In this paper we consider frame properties for systems of functions generated by combined action of dilation, translation and modulation on a function in $L^2(\mathbb{R}^d)$. At the first time, systems of this form were used by A. Cordoba and C. Fefferman in the study of some classes of singular integral operators. They called these systems *wave packet systems*. Wave packet systems have been considered and extended by several authors, see [1], [2], [3], [5], [4].

In the present paper we consider wave packet systems as special cases of *generalized shift-invariant systems,* a concept studied by E. Hernandez, D. Labate and G. Weiss in [3], A. Ron and Z. Shen , as well as Y. Eldar and Christensen . Extending a result in [5], we present a sufficient condition for a generalized shift-invariant system to be a Bessel sequence or even a frame for $L^2(\mathbb{R}^d)$; in particular this leads us to a sufficient condition for a wave packet system to form a frame. Based on a result from [3], we also present some negative results; in fact, certain natural conditions on the parameters in a wave packet system exclude the frame property.
For $y \in \mathbb{R}^d$, the translation operator T_y acting on $f \in L^2(\mathbb{R}^d)$ is defined by

$$(T_y f)(x) = f(x - y), \quad x \in \mathbb{R}^d.$$

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For $y \in \mathbb{R}^d$, the modulation operator E_y is

$$(E_y f)(x) = e^{2\pi i y \cdot x} f(x), \quad x \in \mathbb{R}^d,$$

where $y \cdot x$ denotes the inner product between y and x in $\mathbb{R}^d$. The dilation operator associated with a real $d \times d$ matrix C is

$$(D_C f)(x) = |\det C|^{1/2} f(Cx), \quad x \in \mathbb{R}^d.$$

Definition 1.1. A generalized shift-invariant system is a system of the form

$$\{T_{C_j k} g_j\}_{j \in \mathcal{J}, k \in \mathbb{Z}^d},$$

where $\mathcal{J}$ is a countable collection of indices, $\{g_j\}_{j\in\mathcal{J}} \subset L^2(\mathbb{R}^d)$ and $\{C_j\}_{j\in\mathcal{J}} \subset GL_d(\mathbb{R})$.

Definition 1.2. Let $\{A_j\}_{j\in\mathbb{Z}}$ be a collection of matrices in $GL_d(\mathbb{R})$, let $F \in GL_d(\mathbb{R})$, and let $\{c_m\}_{m\in\mathbb{Z}}$ be a collection of points in $\mathbb{R}^d$. Given a function $g \in L^2(\mathbb{R}^d)$, we will consider the system of functions

$$\{D_{A_j} T_{Fk} E_{c_m} g : j, m \in \mathbb{Z}, k \in \mathbb{Z}^d\}. \tag{1.1}$$

We will also call a system of the type (1.1) a *wave packet system.*

Note that

$$D_{A_j} T_{Fk} E_{c_m} g = T_{A_j^{-1} Fk} D_{A_j} E_{c_m} g. \tag{1.2}$$

By changing the order of dilation, translation and modulation in (1.1), we obtain new systems, but in this paper we only discuss wave packet systems of the form (1.1). The interested reader can find other reorderings of this system in [5].

Definition 1.3. Let I be a countable set and $\mathcal{H}$ a Hilbert space. We say that a sequence $\{f_i\}_{i\in I}$ of members of $\mathcal{H}$ is a frame for $\mathcal{H}$ with frame lower bound A and upper bound B if for each $f \in \mathcal{H}$,

$$A\|f\|^2 \le \sum_{i\in I} |\langle f, f_i\rangle|^2 \le B\|f\|^2.$$

2. Results

In [5], it is proved that if

$$B := \sum_{k\in\mathbb{Z}^d} \sup_{\gamma\in\mathbb{R}^d} \sum_{j\in\mathcal{J}} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)\hat{g}_j(\gamma - C_j^\sharp k)| < \infty$$

and

$$\begin{aligned} A : \ &= \inf_{\gamma\in\mathbb{R}^d} \sum_{j\in\mathcal{J}} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)|^2 \\ &\quad - \sum_{k\neq 0} \sup_{\gamma\in\mathbb{R}^d} \sum_{j\in\mathcal{J}} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)\hat{g}_j(\gamma - C_j^\sharp k)| > 0, \end{aligned} \tag{2.1}$$

then $\{T_{C_j k} g_j\}_{j\in\mathcal{J}, k\in\mathbb{Z}^d}$ is a frame with bounds A and B. The following result is an improvement:

Theorem 2.1. *Let $\{g_j\}_{j\in\mathcal{J}} \subset L^2(\mathbb{R}^d)$, $\mathcal{J}$ countable and $\{C_j\}_{j\in\mathcal{J}} \subset GL_d(\mathbb{R})$. If*

$$B := \sup_{\gamma\in\mathbb{R}^d} \sum_{j\in\mathcal{J}} \sum_{k\in\mathbb{Z}^d} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)\hat{g}_j(\gamma - C_j^{\sharp}k)| < \infty$$

then the generalized shift-invariant system $\{T_{C_jk}g_j\}_{j\in\mathcal{J},k\in\mathbb{Z}^d}$ is a Bessel sequence with bound B. Further, if also

$$A := \inf_{\gamma\in\mathbb{R}^d} \left(\sum_{j\in\mathcal{J}} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)|^2 - \sum_{j\in\mathcal{J}} \sum_{0\neq k\in\mathbb{Z}^d} \frac{1}{|\det C_j|} |\hat{g}_j(\gamma)\hat{g}_j(\gamma - C_j^{\sharp}k)| \right) > 0$$

then $\{T_{C_jk}g_j\}_{j\in\mathcal{J},k\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A and B.

Theorem 2.1 has direct implications for wave packet systems.

Corollary 2.2. *Let $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, $F \in GL_d(\mathbb{R})$, $\{c_m\}_{m\in\mathbb{Z}} \subseteq \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$. Assume that*

$$B := \frac{1}{|\det F|} \sup_{\gamma\in\mathbb{R}^d} \sum_{j,m\in\mathbb{Z}} \sum_{k\in\mathbb{Z}^d} |\hat{g}(A_j^{\sharp}\gamma - c_m)\hat{g}(A_j^{\sharp}\gamma - c_m - F^{\sharp}k)| < \infty. \tag{2.2}$$

Then the wave packet system $\{D_{A_j}T_{Fk}E_{c_m}g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a Bessel sequence with bound B. Further, if also

$$\begin{aligned} A &:= \frac{1}{|\det F|} \inf_{\gamma\in\mathbb{R}^d} \Bigg(\sum_{j,m\in\mathbb{Z}} |\hat{g}(A_j^{\sharp}\gamma - c_m)|^2 \\ &- \sum_{0\neq k\in\mathbb{Z}^d} \sum_{j,m\in\mathbb{Z}} |\hat{g}(A_j^{\sharp}\gamma - c_m)\hat{g}(A_j^{\sharp}\gamma - c_m - F^{\sharp}k)| \Bigg) > 0, \end{aligned}$$

then the wave packet system $\{D_{A_j}T_{Fk}E_{c_m}g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is a frame for $L^2(\mathbb{R}^d)$ with bounds A and B.

Theorem 2.3. *Let $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R}), F \in GL_d(\mathbb{R})$ and assume that there exists a number $C > 0$ and an infinite index set $J \subseteq \mathbb{Z}$ such that*

$$\|A_j\| \leq C, \ \forall j \in I. \tag{2.3}$$

Assume that $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$ satisfies $\bigcup_{m\in\mathbb{Z}}(c_m + B(r,0)) = \mathbb{R}^d$. Then no function $g \in L^2(\mathbb{R}^d)$ satisfying $|\hat{g}(\gamma)| > \epsilon > 0$ can generate a Bessel sequence

$$\{D_{A_j}T_{Fk}E_{c_m}g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}.$$

Corollary 2.4. *Let $\{c_m\}_{m\in\mathbb{Z}} \subset \mathbb{R}^d$ and $g \in L^2(\mathbb{R}^d)$ be as in Theorem 2.3. Furthermore, let $F \in GL_d(\mathbb{R})$ and A be an expansive matrix. Then $\{D_{A^j}T_{Fk}E_{c_m}g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ can not be a Bessel sequence.*

Corollary 2.5. *Let $g \in L^2(\mathbb{R}^d)$ and F be as in Theorem 2.3, let $\{A_j\}_{j\in\mathbb{Z}} \subset GL_d(\mathbb{R})$, and assume that there exist two constants $C, D > 0$ and an infinite index set $I \subseteq \mathbb{Z}$ such that*

$$C \leq \|A_j\| \leq D, \forall j \in I. \tag{2.4}$$

Then the wave packet system $\{D_{A_j}T_{Fk}E_{c_m}g\}_{j,m\in\mathbb{Z},k\in\mathbb{Z}^d}$ is not a Bessel sequence for $L^2(\mathbb{R}^d)$, regardless of the choice of the sequence c_m.

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PROJECTIVE AND INDUCTIVE LIMITS IN LOCALLY CONVEX CONES

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ABSTRACT. We define and study the projective and inductive limit notions for locally convex cones.We use convex quasiuniform structure method for this purpose. Also we study the barreledness in the locally convex cones and introduce the notion upper-barreled cones and prove that the inductive limit of upper-barreled cones is upper-barreled.

1. INTRODUCTION

The general theory of locally convex cones, as developed in [2], deals with preordered cones. We review some of the main concepts and refer to [2] for details.

A *cone* is a set $\mathcal{P}$ endowed with an addition and a scalar multiplication for nonnegative real numbers. The addition is associative and commutative, and there is a neutral element $0 \in \mathcal{P}$. For the scalar multiplication, the usual associative and distributive properties hold. We have $1a = a$ and $0a = 0$ for all $a \in \mathcal{P}$. A *preordered cone* (*ordered cone*) is a cone with a preorder, that is a reflexive transitive relation $\leq$ which is compatible with the algebraic operations.

The extended real numbers $\overline{\mathbb{R}} = \mathbb{R} \cup \{+\infty\}$ is a natural example of an ordered cone with the usual order and algebraic operations in $\overline{\mathbb{R}}$, in particular $0 \cdot (+\infty) = 0$.

A *linear functional* on an ordered cone $\mathcal{P}$ is a mapping $\mu : \mathcal{P} \to \overline{\mathbb{R}}$ such that $\mu(a + b) = \mu(a) + \mu(b)$ and $\mu(\alpha a) = \alpha\mu(a)$ for all $a, b \in \mathcal{P}$ and $\alpha \geq 0$. More generally, for cones $\mathcal{P}$ and $\mathcal{Q}$, a mapping $t : \mathcal{P} \to \mathcal{Q}$ is called a *linear operator* if $t(a + b) = t(a) + t(b)$ and $t(\alpha a) = \alpha t(a)$ hold for $a, b \in \mathcal{P}$ and $\alpha \geq 0$.

A subset $\mathcal{V}$ of the preordered cone $\mathcal{P}$ is called an *(abstract) 0-neighborhood system*, if the following properties hold:

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(i) $0 < v$ for all $v \in \mathcal{V}$;

(ii) for all $u, v \in \mathcal{V}$ there is a $w \in \mathcal{V}$ with $w \leq u$ and $w \leq v$;

(iii) $u + v \in \mathcal{V}$ and $\alpha v \in \mathcal{V}$ whenever $u, v \in \mathcal{V}$ and $\alpha > 0$.

For every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we define

$$v(a) = \{b \in \mathcal{P} \mid b \leq a + v\}, \quad \text{resp.} \quad (a)v = \{b \in \mathcal{P} \mid a \leq b + v\},$$

to be a neighborhood of a in the *upper*, resp. *lower* topologies on $\mathcal{P}$. Their common refinement is called *symmetric* topology. We denote the neighborhoods of the symmetric topology as $v(a) \cap (a)v$ or $v(a)v$ for $a \in \mathcal{P}$ and $v \in \mathcal{V}$. We call $(\mathcal{P}, \mathcal{V})$ a *full locally convex cone*, and each subcone of $\mathcal{P}$, not necessarily containing $\mathcal{V}$, is called a *locally convex cone*. For technical reasons we require the elements of a locally convex cone to be *bounded below*, i.e. for every $a \in \mathcal{P}$ and $v \in \mathcal{V}$ we have $0 \leq a + \rho v$ for some $\rho > 0$. An element a of $(\mathcal{P}, \mathcal{V})$ is called bounded if it is also *upper bounded*, i.e. for every $v \in \mathcal{V}$ there is a $\rho > 0$ such that $a \leq \rho v$. On $\mathcal{P}$ we define the *global preorder* $\preceq$ as follows: $a \preceq b$ if and only if $a \leq b + v$ for all $v \in \mathcal{V}$.

Let $\mathcal{P}$ be a cone. A collection $\mathfrak{U}$ of convex subsets $U \subset \mathcal{P}^2 = \mathcal{P} \times \mathcal{P}$ is called a *convex quasiuniform structure* on $\mathcal{P}$, if the following properties hold:

$(U1)$ $\triangle \subset U$ for every $U \in \mathfrak{U}$ ($\triangle = \{(a, a) : a \in \mathcal{P}\}$);
$(U2)$ for all $U, V \in \mathfrak{U}$ there is a $W \in \mathfrak{U}$ such that $W \subseteq U \cap V$;
$(U3)$ $\lambda U \circ \mu U \subseteq (\lambda + \mu)U$ for all $U \in \mathfrak{U}$ and $\lambda, \mu > 0$;
$(U4)$ $\lambda U \in \mathfrak{U}$ for all $U \in \mathfrak{U}$ and $\lambda > 0$.

Here, for $U, V \subseteq \mathcal{P}^2$, by $U \circ V$ we mean the set of all $(a, b) \in \mathcal{P}^2$ such that there is a $c \in \mathcal{P}$ with $(a, c) \in U$ and $(c, b) \in V$. We call $(\mathcal{P}, \mathfrak{U})$ a *convex quasiuniform cone.*

To every convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ we associate a preorder defined by $a \leq b$ if and only if $(a, b) \in U$ for all $U \in \mathfrak{U}$ and, two topologies: The neighborhood bases for an element a in the upper and lower topologies are given by the sets

$$U(a) = \{b \in \mathcal{P} : (b, a) \in U\}, \quad \text{resp.} \quad (a)U = \{b \in \mathcal{P} : (a, b) \in U\}, \quad U \in \mathfrak{U}.$$

The topology associated with the uniform structure $\mathfrak{U}_s = \{U \cap U^{-1} : U \in \mathfrak{U}\}$ is the common refinement of the upper and lower topologies, where $U^{-1} = \{(b, a) \ : \ (a, b) \in U\}$.

Let $(\mathcal{P}, \mathfrak{U})$ and $(\mathcal{Q}, \mathfrak{V})$ be two convex quasiuniform cones and $t : \mathcal{P} \to \mathcal{Q}$ be a linear mapping. We say that t is *uniformly continuous* if for each $V \in \mathfrak{V}$, there is a $U \in \mathfrak{U}$ such that $(a, b) \in U$ implies $(t(a), t(b)) \in V$ or $T(U) \subseteq V$, $T = t \times t$. Let $\mathfrak{U}_1$ and $\mathfrak{U}_2$ be convex quasiuniform structures on $\mathcal{P}$. Following N. Bourbaki, [1], II, 2.2, we say that $\mathfrak{U}_1$ is *finer* than $\mathfrak{U}_2$ if the identity mapping $i : (\mathcal{P}, \mathfrak{U}_1) \to (\mathcal{P}, \mathfrak{U}_2)$ is uniformly continuous.

The notions of an (abstract) 0-neighborhood system $\mathcal{V}$ and a convex quasiuniform structure $\mathfrak{U}$ for a cone $\mathcal{P}$ are equivalent in the following sense:

Let $\mathcal{P}$ be a preordered cone and $\mathcal{V}$ an (abstract) 0-neighborhood system for $\mathcal{P}$. For each $v \in \mathcal{V}$, we put

$$\tilde{v} = \{(a, b) \in \mathcal{P} \times \mathcal{P} \ : \ a \leq b + v\}.$$

The collection $\widetilde{\mathcal{V}} = \{\tilde{v} : v \in \mathcal{V}\}$ is a convex quasiuniform structure on $\mathcal{P}$, which induces the global preorder on $\mathcal{P}$ and the same upper, lower and symmetric topologies. Furthermore, if $(\mathcal{P}, \mathcal{V})$ is a locally convex cone, i.e. each element of $\mathcal{P}$ is bounded below, we have:

$$\text{For all } a \in \mathcal{P} \text{ and } \tilde{v} \in \widetilde{\mathcal{V}} \text{ there is some } \rho > 0 \text{ such that } (0, a) \in \rho\tilde{v}.$$

On the other hand:

If $\mathcal{P}$ is a cone with a convex quasiuniform structure $\mathfrak{U}$, then one can find a preorder and an (abstract) 0-neighborhood system $\mathcal{V}$ such that the convex quasiuniform structure $\widetilde{\mathcal{V}}$ is equivalent to $\mathfrak{U}$ (see [2], I.5.5). In this case if we also have

$$(U5) \text{ for all } a \in \mathcal{P} \text{ and } U \in \mathfrak{U} \text{ there is some } \rho > 0 \text{ such that } (0, a) \in \rho U,$$

then by the equivalency of $\mathfrak{U}$ and $\widetilde{\mathcal{V}}$, all elements of $\mathcal{P}$ would be bounded below. $(U1) - (U5)$ make $(\mathcal{P}, \mathfrak{U})$ into a locally convex cone.

The u-continuous linear functionals on a locally convex cone $(\mathcal{P}, \mathcal{V})$ (into $\overline{\mathbb{R}}$) form a cone with the usual addition and scalar multiplication of functions. This cone is called the *dual cone* of $\mathcal{P}$ and denoted by $\mathcal{P}^*$.

For a locally convex cone $(\mathcal{P}, \mathcal{V})$, the polar v° of $v \in \mathcal{V}$ consists of all linear functionals μ on $\mathcal{P}$ satisfying $\mu(a) \le \mu(b) + 1$ whenever $a \le b + v$ for $a, b \in \mathcal{P}$. We have $\cup\{v^\circ : v \in \mathcal{V}\} = \mathcal{P}^*$.

2. Projective limits

Theorem 2.1. *For each $\gamma \in \Gamma$, let $\mathcal{P}_\gamma$ be a cone with a convex quasiuniform structure $\mathfrak{U}_\gamma = \{U_{\gamma\delta} : \delta \in \mathcal{D}_\gamma\}$. Let $\mathcal{P}$ be a cone and, for each $\gamma \in \Gamma$, g_γ be a linear mapping of $\mathcal{P}$ into $\mathcal{P}_\gamma$. Then there is a coarsest convex quasiuniform structure $\mathfrak{U}$ on $\mathcal{P}$ under which all the g_γ are u-continuous. If all $\mathcal{P}_\gamma$'s are locally convex, then $\mathcal{P}$ is also locally convex.*

The locally convex cone $\mathcal{P}$ with the preorder and (abstract) 0-neighborhood system induced by this convex quasiuniform structure is called the *projective limit* of the locally convex cones $\mathcal{P}_\gamma$ by the mappings g_γ.

For $a \in (\mathcal{P}, \mathcal{V})$, we define $\bar{a} = \cap\{v(a) : v \in \mathcal{V}\}$, and we call $\mathcal{P}$ *separated* if $\bar{a} = \bar{b}$ implies $a = b$ for all $a, b \in \mathcal{P}$.

Proposition 2.2. *The locally convex cone $\mathcal{P}$ is separated if and only if the symmetric topology on $\mathcal{P}$ is Hausdorff.*

Let $\{g_\gamma : \gamma \in \Gamma\}$ be a family of functions on a cone $\mathcal{P}$. We say that $\{g_\gamma : \gamma \in \Gamma\}$ is a *separating* family of functions over $\mathcal{P}$, if whenever $x_1 \ne x_2$, there is a g_γ $(\gamma \in \Gamma)$ such that $g_\gamma(x_1) \ne g_\gamma(x_2)$.

Proposition 2.3. *Let $(\mathcal{P}, \mathcal{V})$ be the projective limit of the locally convex cones $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ by the mappings g_γ, $\gamma \in \Gamma$. If each $\mathcal{P}_\gamma$ is separated and $\{g_\gamma : \gamma \in \Gamma\}$ is separating, then $\mathcal{P}$ is separated.*

We define a subset $A \subseteq \mathcal{P} = (\mathcal{P}, \mathfrak{U})$ to be *bounded* if for every $U \in \mathfrak{U}$ there is a $\lambda_U > 0$ such that

$$(0, a)\ ,\ (a, 0) \in \lambda_U U \quad \text{for all} \quad a \in A.$$

A subset A of locally convex cone $(\mathcal{P}, \mathcal{V})$ is called *precompact* with respect to the symmetric topology if for every $v \in \mathcal{V}$, there are $a_1, \ldots, a_n \in A$ such that $A \subseteq$

$\cup_{i=1}^{n} v(a_i)v$. If t is a u-continuous linear mapping of $\mathcal{P}$ into locally convex cone $(\mathcal{Q}, \mathcal{W})$ and $A \subseteq \mathcal{P}$ is precompact, then $t(A)$ is also precompact.

Proposition 2.4. *Let $(\mathcal{P}, \mathcal{V})$ be the projective limit of the locally convex cones $(\mathcal{P}_\gamma, \mathcal{V}_\gamma)$ by the mappings g_γ. Then the subset A of $\mathcal{P}$ is bounded, or precompact, if and only if each $g_\gamma(A)$ has the same property.*

3. Inductive limits

Theorem 3.1. *For each $\gamma \in \Gamma$ let $\mathcal{P}_\gamma$ be a cone with a convex quasiuniform structure $\mathfrak{U}_\gamma$. Let $\mathcal{P}$ be a cone and, for each $\gamma \in \Gamma$, $f_\gamma : \mathcal{P}_\gamma \to \mathcal{P}$ is a linear mapping such that $\mathcal{P} = span \bigcup_{\gamma \in \Gamma} f_\gamma(\mathcal{P}_\gamma)$. Let $\mathfrak{U}$ be the set of all convex subsets of $\mathcal{P}^2$ such that:*

(i) *for each $U \in \mathfrak{U}$, and each $\gamma \in \Gamma$, we have $F_\gamma^{-1}(U) \in \mathfrak{U}_\gamma$.*
(ii) *Each $U \in \mathfrak{U}$ satisfies (U3).*
(iii) *For every $U_1, \ldots, U_n \in \mathfrak{U}$ we have $U_1 \cap \ldots \cap U_n \in \mathfrak{U}$.*

Then $\mathfrak{U}$ is the finest quasiuniform structure on $\mathcal{P}$ which makes each f_γ u-continuous. If all $\mathcal{P}_\gamma$'s are locally convex, then $\mathcal{P}$ is also locally convex.

Here $F_\gamma = f_\gamma \times f_\gamma$; also, by linearity and span we mean the linearity and span on non-negative scalars only.

The locally convex cone $\mathcal{P}$ with the preorder and (abstract) 0-neighborhood system induced by this convex quasiuniform structure is called the *inductive limit* of the locally convex cones $\mathcal{P}_\gamma$ by the mappings f_γ.

We call the locally convex cone $(\mathcal{P}, \mathcal{V})$ *bornological* if each bounded linear mapping from $\mathcal{P}$ to an arbitrary locally convex cone is u-continuous.

Proposition 3.2. *An inductive limit of bornological cones is bornological.*

4. barreledness

Definition 4.1. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone. A *barrel* is a convex subset B of $\mathcal{P}^2$ with the following properties :

$(B1)$ For every $b \in \mathcal{P}$ there is a $v \in \mathcal{V}$ such that for every $a \in v(b)v$ there is a $\lambda > 0$ such that $(a, b) \in \lambda B$.
$(B2)$ For all $a, b \in \mathcal{P}$ such that $(a, b) \notin B$ there is a $\mu \in \mathcal{P}^*$ such that $\mu(c) \leq \mu(d)+1$ for all $(c, d) \in B$ and $\mu(a) > \mu(b) + 1$.

Theorem 4.2. *In a locally convex cone $(\mathcal{P}, \mathcal{V})$, the set of all barrels $\mathfrak{B}$ is a convex quasiuniform structure on $\mathcal{P}$.*

A locally convex cone $(\mathcal{P}, \mathcal{V})$ is said to be *tightly covered by its bounded elements* if for all $a, b \in \mathcal{P}$ and $v \in \mathcal{V}$ and $a \notin v(b)$ (or a not less than or equal to $b + v$) there is some bounded element $a' \in \mathcal{P}$ such that $a' \preceq a$ and $a' \notin v(b)$.

Proposition 4.3. *If locally convex cone $(\mathcal{P}, \mathcal{V})$ is tightly covered by its bounded elements, then $\tilde{v} = \{(a, b) \in \mathcal{P}^2 \ : \ a \leq b + v\}$ is a barrel for every $v \in \mathcal{V}$.*

Definition 4.4. Let $(\mathcal{P}, \mathcal{V})$ be a locally convex cone and $\widetilde{\mathcal{V}}$ be the convex quasiuniform structure generated by $\mathcal{V}$. $\mathcal{P}$ is called *upper-barreled* if for every barrel $B \subseteq \mathcal{P}^2$, there is a $\tilde{v} \in \widetilde{\mathcal{V}}$ such that $\tilde{v} \subseteq B$.

Theorem 4.5. *An inductive limit of upper-barreled cones is upper-barreled.*

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SUBADDITIVE SEPARATING MAPS BETWEEN REGULAR BANACH FUNCTION ALGEBRAS

FERESHTEH SADY [1]* AND YOUSEF ESTAREMI [2]

Abstract. In this note we extend the results of E. Beckenstein and L. Narici concerning subadditive separating maps from $A = C(X)$ to $B = C(Y)$, for compact Hausdorff spaces X and Y, to the case where A and B are regular Banach function algebras (not necessarily unital) with A satisfying Ditkin's condition.

1. Introduction and preliminaries

Let A and B be two spaces of functions. A map $H : A \longrightarrow B$ is called *separating* if $f.g = 0$ implies $H(f).H(g) = 0$, for all $f, g \in A$. Weighted composition operators are important typical examples of separating maps. Clearly any algebra homomorphism between algebras of functions is separating. Moreover, if A and B are both lattices then every lattice homomorphism is a separating map. The study of separating maps between different spaces of functions has attracted a considerable interest in recent years. The general form of linear separating maps between algebras of continuous functions on compact Hausdorff spaces were considered in [5]. Later in [4] the results were extended to certain regular Banach function algebras. For a recent survey on this topic one can refer to [1]. On the other hand in [2] the well known results concerning linear separating maps from $C(X)$ to $C(Y)$, for compact Hausdorff spaces X and Y, were extended to not necessarily linear case. In this paper we revisit the results of [2] and extend some of these results to a more general case. In particular we describe the general form of subadditive separating maps between regular Banach function algebras.

For a commutative Banach algebra A, $m(A)$ denotes its maximal ideal space and for an element x in A, $\hat{x} \in C_0(m(A))$ is its Gelfand transform. For definitions and well-known results related to Banach function algebras we refer the reader to [3]. The

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* Speaker.

notation δ_x will be used for the evaluation homomorphism at some point $x \in X$ of a Banach function algebra on a locally compact Hausdorff space X.

2. Main results

Throughout this section $(A, \|\cdot\|_A)$ and $(B, \|\cdot\|_B)$ are Banach function algebras on their maximal ideal spaces X and Y, respectively. Let X_∞ and Y_∞ be the one-point compactifications of X and Y, respectively. Given an element $f \in A \subseteq C_0(X)$ let $\mathrm{coz}(f)$ denote the cozero set of f. For a separating map $H : A \longrightarrow B$ (without any linearity assumption) let us define the set Y_0 to be $\{y \in Y : Hf(y) \neq 0, \text{ for some } f \in A\}$. Clearly Y_0 is an open subset of Y and since H is separating, $H0 = 0$ on Y_0. The standard tools in dealing with separating maps are vanishing sets defined in the following. The definition in the case of net necessarily linear case is the same as in the linear case.

Definition 2.1. Let $y \in Y_0$. An open subset U of X_∞ is called a *vanishing set* for $\delta_y \circ H$, if for each $f \in A$, $\mathrm{coz}(f) \subseteq U$ implies that $\delta_y \circ H(f) = 0$. The *support* of $\delta_y \circ H$ is then defined by

$$\mathrm{supp}\ \delta_y \circ H = X_\infty \setminus \cup \{V \subseteq X_\infty : V \text{ is a vanishing set for } \delta_y \circ H\}.$$

Using the partition of unity theorem for regular Banach function algebras, we see that for a certain separating map H, $\mathrm{supp}\ \delta_y \circ H$ is a singleton, for all $y \in Y_0$. Before stating our results we need the following definition borrowed from [2].

Definition 2.2. A map $T : A \longrightarrow B$ is called *pointwise subadditive* if for all $f, g \in A$ and $y \in Y$, $|T(f+g)(y)| \leq |Tf(y)| + |Tg(y)|$.

Theorem 2.3. *If A is regular and $H : A \longrightarrow B$ is a pointwise subadditive separating map, then for all $y \in Y_0$ the set $\mathrm{supp}\, \delta_y \circ H$ is a singleton.*

Definition 2.4. Under the hypothesis of the preceding theorem we can correspond to each $y \in Y_0$ an element $h(y) \in X_\infty$ which is the unique point of $\mathrm{supp}\ \delta_y \circ H$. We call the mapping $h : Y_0 \longrightarrow X_\infty$ the *support map* of H.

The following results can also be obtained with minor modifications of [2, Theorem 4.3].

Theorem 2.5. *Let A and B be regular and $H : A \longrightarrow B$ be a pointwise subadditive separating map. Then*

a) $h(\mathrm{coz}(Hf)) \subseteq cl_{X_\infty}(\mathrm{coz}(f))$, *for all* $f \in A$,

b) $\{h(y)\} = \bigcap_{\delta_y \circ H(f) \neq 0} cl_{X_\infty}(\mathrm{coz}(f))$, $y \in Y_0$.

The following concept was introduced in a special case in [2].

Definition 2.6. Let A be regular and $H : A \longrightarrow B$ be a pointwise subadditive separating map with the support map h. Then we call H *strongly pointwise subadditive* if for each $y \in Y_0$ there exist $M_y > 0$ such that for each scalar c

$$|Hf(y) - Hg(y)| \leq M_y |H(f-g)(y)|$$

holds for all $f, g \in C(X)$ with $f(h(y)) = c$ and $|f(h(y)) - g(h(y))| < \delta_{c,y}$, for some scalar $\delta_{c,y}$ depending only on c and y.

Lemma 2.7. *Let A and B be regular and $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. Then*

a) *if $f, g \in A$ and $f = g$ on a neighborhood U in X_∞, then $Hf = Hg$ on $h^{-1}(U)$.*
b) *the support map $h : Y_0 \longrightarrow X_\infty$ of H is continuous.*
c) *if h is injective then $h(Y_0)$ is dense in X_∞.*

Theorem 2.8. *Let A and B be regulars and $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. If A satisfies Ditkin's condition then $h(y) \in X$, when $y \in Y_0$ is such that $\delta_y \circ H$ is $\| \cdot \|_A$-continuous.*

Let A and B regular and $H : A \longrightarrow B$ be a pointwise subadditive separating map. Assume $y \in h^{-1}(X)$ and U is an open neighborhood of $h(y)$ in X with compact closure in X. By regularity of A we can choose an element $e_{y,U} \in A$ such that $e_{y,U} = 1$ on U. It follows from Lemma 2.7(a) that for each scalar α, $H(\alpha e_{y,U})(y)$ is independent on the choice of U and $e_{y,U}$.

Theorem 2.9. *Let A and B be regular where A satisfies Ditkin's condition and let $H : A \longrightarrow B$ be a strongly pointwise subadditive separating map. If $y \in Y_c := \{y \in Y_0 : \delta_y \circ H$ is $\| \cdot \|_A -$ continuous$\}$ then $Hf(y) = H(f(h(y)).e_{y,U})(y)$, for all $f \in A$.*

Remark 2.10. The above result has been proved in [4, Proposition 5] when A is just regular and H is linear with the set Y_c defined by $Y_c = \{y \in Y : \delta_y \circ H$ is $\| \cdot \|_\infty -$ continuous$\}$. But in the proof of Proposition 4 in [4] there is a little gap. The sequence $\{g_n\}$ in A given in this proof, has no common bound so that the convergence $\|fg_n - f\|_\infty \longrightarrow 0$ is not clear. Any way Corollary 2.12 below shows that if A is regular and satisfies Ditkin's condition and moreover $H : A \longrightarrow B$ is a linear separating map then for each $y \in Y_0$, $\delta_y \circ H$ is $\|.\|_\infty$ continuous iff it is $\|.\|_A$ continuous.

The following definition is an extension of the concept "1-boundedness" defined in [2]to the case of not necessarily unital Banach function algebras.

Definition 2.11. We call a separating map $H : A \longrightarrow B$ *locally 1-bounded*, if there exists a scalar $D > 0$ such that for each $y \in Y$ we can choose a neighborhood U of y and an appropriate $e_{y,U} \in A$ ($e_{y,U} = 1$ on U) such that $\|H(ae_{y,U})\|_B \leq D|a|.\|H(e_{y,U})\|_B$, for all scalars a.

Theorem 2.12. *Let A and B be as in Theorem 2.9. If H is locally 1-bounded and $y \in Y_0$ then $y \in Y_c$ if and only if $Hf(y) = H(f(h(y)).e_{y,U})(y)$ holds for all $f \in A$*

Corollary 2.13. *Let A, B and H be as in Theorem 2.12. Then for $y \in Y_0$, $\delta_y \circ H$ is $\|.\|_A$ continuous iff $\delta \circ H$ is $\|.\|_\infty$ continuous.*

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A COUNTING METHOD FOR APPROXIMATING $\zeta(n)$

MOHAMMAD HASSAN SHIRDARREH HAGHIGHI

Abstract. In this paper, we give a counting method for approximating $\zeta(n)$, for every natural number $n > 1$, where ζ denotes the Riemann zeta function and defined by

$$\zeta(s) = \Sigma_{k=1}^{\infty} \frac{1}{k^s} \qquad (s > 1).$$

1. Introduction and preliminaries

The Riemann Zeta function is an important special function in mathematics. It is defined by the series

$$(*) \qquad \zeta(s) = \Sigma_{k=1}^{\infty} \frac{1}{k^s},$$

for every real number $s > 1$ and can be extended analytically to all complex numbers $s \neq 1$. There exist many open problems concerning zeta function ([1]); the most famous of them is the Riemann hypothesis about the distribution of its zeros.

The Zeta function relates to the prime numbers through the Euler product equality ([1]):

$$\zeta(s) = [\Pi_{k=1}^{\infty}(1 - {p_k}^{-s})]^{-1},$$

where p_k is the kth prime number, and this equality makes it important in analytic number theory. To approximate $\zeta(n)$ for different values of n, we can use the summation formula $(*)$. Also there are many equivalent formulas such as

$$\zeta(s) = \frac{1}{\Gamma(s)} \int_0^{\infty} \frac{u^{s-1}}{e^u - 1} du,$$

for real $s > 1$.

But after demonstrating the main theorem, which gives $\zeta(n)$ in terms of the probability that the greatest common divisor (gcd) of an n-tuple of natural numbers be 1,

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we can estimate this probability and then approximate the values of $\zeta(n)$, for natural numbers $n > 1$. On the other hand if $\zeta(n)$ is known, q_n can easily be computed.

The values of $\zeta(n)$ can be computed exactly for even numbers by Fourier analysis methods, though it is known exactly for no odd number. The only known result on odd numbers is a result due to Apéry which says that $\zeta(3)$ is an irrational number, so $\zeta(3)$ is called the Apéry's constant ([1]).

2. Main result

Theorem. Let q_n denote the probability that $\gcd(a_1, \ldots, a_n) = 1$ for n-tuples of natural numbers, $n \geq 2$. Then $q_n = 1/\zeta(n)$.

The probability q_n, on an infinite sample space, is an asymptotic probability, but to estimate it, we can use large finite sets of n-tuples of natural numbers. We begin with an integer i and end to an integer j, counting how many n-tuples with components between i and j, have gcd=1. To determine the gcd of an n-tuple we use the Euclid dividing algorithm. This is done by a computer program written in Delphi 7 language by my student Aref Rowshan which takes the initial and final values i and j, counts how many n-tuples with components between i and j are co-prime and divides this count to the number of all possible n-tuples in that range. After all we have an estimation of q_n or equivalently for $\zeta(n)$.

The estimations of $\zeta(n)$, for $n = 2, 3, 4$ with the given values of i and j, are in the following table, comparing with exact values to ten decimals.

n	i	j	estimated $\zeta(n)$	exact $\zeta(n)$ to ten decimals
2	10	100	1.6641881028	1.64493406684
2	1	100	1.6425755584	”
2	1	10	1.5624999998	”
2	1	1000	1.6436987166	”
3	1	100	1.1912567500	1.2020569032
3	10	100	1.1970852548	”
3	1	1000	1.2008874890	”
3	100	1000	1.2014203594	”
4	1	10	1.1421725239	1 .0823232323
4	10	100	1.0926743011	”
4	1	100	1.0883363496	”

The method gives an interesting estimation of π, because $\zeta(2) = \pi^2/6$.

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TOPOLOGICAL ACTION OF A UNITAL INVERSE SEMIGROUP

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ABSTRACT. Given a unital inverse semigroup S, we define the *topological action* θ of S on a locally compact Hausdorff space X. Also an action α of S on the C^*-algebra $C_0(X)$ *corresponding to* θ is introduced. Topologically freeness of θ and the effect of θ on α is considered. Finally the notion of *semipartial dynamical system* $(C_0(X), S, \alpha)$ and some of its properties are discussed.

1. INTRODUCTION

By a *unital inverse semigroup* we mean a semigroup S with the unit element e such that for each s in S, there exists a unique element s^* in S with the following properties:
(i) $ss^*s = s$;
(ii) $s^*ss^* = s^*$.
Let A be a C^*-algebra. A *partial automorphism* of A is a triple (α, I, J), where I and J are closed two-sided ideals in A and $\alpha : I \to J$ is a $*$-isomorphism.

For given partial automorphisms (α, I, J) and (β, K, L) of A, their product $\alpha\beta$ is nothing but the composition of α and β with the largest possible domain, that is, $\alpha\beta : \beta^{-1}(I) \to A$ such that $(\alpha\beta)(a) = \alpha(\beta(a))$. Obviously, $\beta^{-1}(I)$ is a closed ideal of K and since ideals of ideals of a C^*-algebra are, themselves, ideals of that algebra, the product $(\alpha\beta, \beta^{-1}(I), \alpha\beta(\beta^{-1}(I)))$ is also a partial automorphism. It is not hard to see that the set $P\mathrm{Aut}(\mathrm{A})$ of partial automorphisms of A is a unital inverse semigroup under the composition with the largest passible domain with the identity (i, A, A), where i is the identity map on A and $(\alpha, I, J)^* = (\alpha^{-1}, J, I)$.

Together with the notion of partial actions of groups on C^*-algebras, a generalization of the concept of crossed product of C^*-algebras appeared in the theory of operator algebras (see [1], [2], [3], [4], [5]). Following their footsteps, the definition

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of an action of a unital inverse semigroup on a C^*-algebra is given. Also the crossed product of a C^*-algebra and a unital inverse semigroup is introduced.

Definition 1.1. Let A be a C^*-algebra and S be a unital inverse semigroup with the identity e. An *action* of S on A is a semigroup homomorphism $s \longmapsto (\alpha_s, E_{s^*}, E_s) : S \to P\mathrm{Aut}(\mathrm{A})$, with $E_e = A$. An element s of S is called idempotent if $s^2 = s$. And S is called an idempotent semigroup if $s^2 = s$ for all s in S.

Lemma 1.2. *Let S be an inverse semigroup, α an action of S on a C^*-algebra A and $s \in S$, then $\alpha_{s^*} = \alpha_s^{-1}, \alpha_e$ is the identity map on A and if s is an idempotent, the α_s is the identity map on $E_{s^*} = E_s$.*

Lemma 1.3. *If α is an action of the unital inverse semigroup S on A, then*

$$\alpha_t(E_{t^*}E_s) = E_{ts}, \forall s, t \in S.$$

Let α be an action of the unital inverse semigroup S on the C^*-algebra A. Consider $L = \{x \in \ell^1(S, A) : x(s) \in E_s\}$, the closed subspace of $\ell^1(S, A)$. Define multiplication and involution on L by

$$(x * y)(s) = \sum_{rt=s} \alpha_r[\alpha_{r^*}(x(r))y(t)] \quad \text{and} \quad x^*(s) = \alpha_s[x(s^*)^*].$$

Note that L is closed with respect to the above operations, since by Lemma 1.3, elements of the form $(x * y)(s)$ are in E_s for every $s \in S$ and as a consequence $x * y \in L$. Also, for given x in L, since $x(s^*) \in E_{s^*}$ and E_{s^*} is an ideal of A we have $(s(x^*))^* \in E_{s^*}$. Therefore $\alpha_s(x(s^*)^*) \in E_s$, so $x^* \in L$. Simple computations show that $||x * y|| \leq ||x||||y||$ and $||x^*|| = ||x||$, where $||.||$ denotes the norm of L inherited from $\ell^1(S, A)$.

If (π, v, H) is a covariant representation of (A, S, α), then $\pi \times v$ is a non-degenerate representation of L.

Definition 1.4. Let A be a C^*-algebra and α be an action of the unital inverse semigroup S on A. The seminorm $||.||_c$ on L is defined by $||x||_c = \sup\{||\pi \times v)(x)|| : (\pi, v)$ is a covariant representation of $(A, S, \alpha)\}$.

Let $I = \{x \in L : ||x||_c = 0\}$. The *crossed product* $A \times_\alpha S$ is the C^*-algebra obtained by completing the quotient L/I with respect to $||.||_c$.

2. Topological Actions

By a *semipartial dynamical system* we mean a triple (A, S, α), in which A is a C^*-algebra, S is a unital inverse semigroup and α is an action of S on A. In this section we will mostly be concerned with $(C_0(X), S, \alpha)$, where X is a locally compact Hausdorff space and α is the action of S on $C_0(X)$, which arises from partial homeomorphisms of X. That is, for every $s \in S$ there is an open subset U_s of X and a homeomorphism $\theta_s : U_{s^*} \to U_s$ such that $U_e = X$ and θ_e is the identity map on X. The action α of S on $C_0(X)$ corresponding to the partial homeomorphism θ is given by

$$\alpha_s(f)(x) = f(\theta_{s^*}(x)), \;\; s \in S, \;\; f \in C_0(U_{s^*}).$$

Now we can summarize the above facts in the following definition.

Definition 2.1. Let S be a unital inverse semigroup and X be a locally compact Hausdorff space. A *topological action of* S *on* X is a pair $\theta = (\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$, where for each s in S, U_s is an open subset of $X, \theta_s : U_{s^*} \to U_s$ is a homeomorphism, $U_e = X$ and θ_e is the identity map on X.

Given a topological action $(\{U_s\}_{s\in S}, \{\theta_s\}_{s\in S})$ of S on a locally compact Hausdorff space X, let $E_s = C_0(U_s)$ be identified, in the usual way, with the ideal of functions in $C_0(X)$ vanishing on $X - U_s$. Therefore we have the following definition.

Definition 2.2. The action α of S on $C_0(X)$ *corresponding to the topological action* θ is given by

$$\alpha_s(f)(x) := f(\theta_{s^*}(x)), f \in C_0(U_{s^*})$$

for each s in S.

Definition 2.3. The topological action θ of S on X is *topologically free* if for every $s \in S - \{e\}$ the set $F_s := \{x \in U_{s^*} : \theta_s(x) = x\}$ has empty interior.

Theorem 2.4. *The topological action θ of a unital inverse semigroup S on X is topologically free if and only if for every $s \in S - \{e\}$, the set F_s is nowhere dense.*

The following equivalent version of topological freeness is more appropriate for our purposes.

Corollary 2.5. *The topological action θ of S on X is topologically free if and only if for every finite subset $\{s_1, s_2, \cdots, s_n\}$ of $S - \{e\}$, the set $\bigcup_{i=1}^{n} F_{s_i}$ has empty interior.*

In the remainder of this section we denote by $\delta_s(s \in S)$ the function in L, which takes the value 1 at s and zero at every other element of S.

Theorem 2.6. *Let $s \in S - \{e\}, f \in E_s = C_0(U_s)$, and $x_0 \notin F_s$. For every $\varepsilon > 0$ there exists $h \in C_0(X)$ such that:*
(i) $h(x_0) = 1$*;*
(ii) $||h(f\delta_s)h|| \leq \varepsilon$*, and*
(iii) $0 \leq h \leq 1$*.*

Definition 2.7. If A is a C^*-algebra and if B is a C^*-subalgebra of A, then by a *conditional expectation* from A to B we mean a continuous positive projection of A onto B, which satisfies the conditional expectation property

$$P(ba) = bP(a) \text{ and } P(ab) = P(a)b, \ b \in B, \ a \in A.$$

We can consider $C_0(X)$ as a C^*-subalgebra of the partial crossed product $C_0(X) \times_\alpha S$. Therefore the conditional expectation from $C_0(X) \times_\alpha S$ onto $C_0(X)$, which is denoted by E is meaningful.

Definition 2.8. A semipartial dynamical system (A, S, α) is said to be *topologically free* if the set of fixed points for the partial homeomorphism associated to each nontrivial semigroup element has empty interior.

It is well known that a crossed product by a partial action is a graded C^*-algebra. Since the conditional expectation $E : C_0(X) \times_\alpha S \to C_0(X)$ is contractive we can state and prove the following theorem.

Theorem 2.9. *If* $(C_0(X), S, \alpha)$ *is a topologically free semipartial dynamical system then for every* $c \in C_0(X) \times_\alpha S$ *and every* $\varepsilon > 0$ *there exists* $h \in C_0(X)$ *such that:*
(i) $||hE(c)h|| \geq ||E(c)|| - \varepsilon$,
(ii) $||hE(c)h - hch|| \leq \varepsilon$, *and*
(iii) $0 \leq h \leq 1$.

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OBLIQUE PREDICTION OF STATIONARY RANDOM FIELDS

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ABSTRACT. Prediction theory of stationary random fields with respect to their oblique past is studied and spectral criteria for determinism, non-determinism and purely non- determinism with respect to these past are obtained.

1. Introduction and preliminaries

In this note we consider a type of past, called *oblique* past and study its prediction problems. Our technique is based on the idea of transforming indices of a given random field, in such a way that it converts our oblique past problem to a horizontal past one regarding the new random field and then invoke the existing horizontal results to solve the problem.

2. Main results

Let $\mathbf{H}$ be the Hilbert space $L_0^2(\Omega, P)$ of all zero-mean random variables on a probability space (Ω, P) which have finite variance. A family $x_{m,n}; m, n \in Z$ of random variables on (Ω, P) is called a stationary second order random field (ssorf, in short) if $\{x_{m,n} : m, n \in Z\}$ is a subset of $\mathbf{H}$ and

$$\langle x_{m,n}, x_{m',n'} \rangle = \Gamma_x(m - m', n - n');\ m, m', n, n' \in Z.$$

It is well known that covariance kernel

$$\Gamma_x(m, n) := \langle x_{m,n}, x_{0,0} \rangle;\ m, n \in Z$$

of any ssorf $x_{m,n}$ admits a spectral representation of the form

$$\Gamma_x(m, n) = \frac{1}{(2\pi)^2} \int_T e^{i(m\lambda + n\theta)} d\nu_x(\lambda, \theta);\ m, n \in Z,$$

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* Speaker.

where the *spectral measure* ν_x of $x_{m,n}$ is a uniquely determined bounded nonnegative measure on $T := [0, 2\pi) \times [0, 2\pi)$.

We need the following terminologies

Definition 2.1. Given a ssorf $x_{m,n}$ we define

- its *horizontal past* at time (m, n) by
$$H(x; m, n) := \overline{sp}\{x_{j,k} : j < m, k \in Z\},$$
- its *augmented horizontal past* at time (m, n) by
$$H^+(x; m, n) := \overline{sp}\{x_{j,k} : j < m \ \text{ or } \ j = m, kn\},$$
- its *horizontal (or augmented horizontal) remote past* by
$$H(x; -\infty) := \cap_{m,n} H(x; m, n)(= \cap_{m,n} H^+(x; m, n)),$$
- and its *time domain* by
$$H(x) := \overline{sp}\{x_{j,k} : j, k \in Z\}.$$

Definition 2.2. A ssorf $x_{m,n}$ is called

- H *-deterministic* if $H(x; -\infty) = H(x)$,
- H^+ *-deterministic* if $H^+(x; -\infty) = H(x)$,
- H *-purely nondeterministic* (or H^+ *-purely nondeterministic*) if
$$H(x; -\infty) = \{0\}.$$

It is easy to see that a ssorf $x_{m,n}$ is H-deterministic(respectively H^+-deterministic) iff

$$x_{0,0} \in H(x; 0, 0)(\text{respectively } x_{0,0} \in H^+(x; 0, 0)).$$

3. Index transform

Throughout this note $x_{m,n}$ stands for a ssorf and $Q(m, n) = (Am + Bn, Cm + Dn)$ for an index-transform on the lattice points Z^2 with integers coefficients A,B,C and D which satisfy $AD - BC = 1$. To each such index-transform Q we associate a transformation L on R^2 given by $L(\lambda, \theta) = (A\lambda + C\theta, B\lambda + D\theta)$ and an index set

$$J(Q) = \{(j, k) \in Z^2 : T_{j,k} \cap L(T) \neq \emptyset\},$$

where for each integers j and k, $S_{j,k} : R^2 \to R^2$ is the shift defined by $S_{j,k}(\lambda, \theta) = (\lambda + 2j\pi, \theta + 2k\pi)$ and $T_{j,k} = S_{j,k}(T)$. By the transform $y_{m,n}$ of $x_{m,n}$ via Q we mean $y = xoQ$ or $y_{m,n} = x_{Am+Bn,Cm+Dn}$.

In the rest of this section we reveal some close ties which exist between any ssorf $x_{m,n}$ and its transform.

Lemma 3.1. *Any transform $y = xoQ$ of any ssorf $x_{m,n}$ is a ssorf.*

Proposition 3.2. *If $x_{m,n}$ is a ssorf with spectral measure ν_x then spectral measure of its transform $y = xoQ$ is given by*

$$\mu_x^Q(E) := \sum_{(j,k)\in J(Q)} \nu_x(L^{-1}(S_{j,k}(E \cap M_{j,k}))), E \subset T, \tag{3.1}$$

where $M_{j,k} = S_{-j,-k}(T_{j,k} \cap L(T))$. In other words $\nu_y = \mu_x^Q$.

Proposition 3.3. *Spectral measure ν_x of a ssorf $x_{m,n}$ is a.c. w.r.t. the Leb. measure if and only if μ_x^Q is so.*

We denote the Radon-Nikodym derivative of the a.c. part of ν_x with respect to the Lebesgue measure by $k_x(\lambda,\theta)$ (i.e. $\nu_x(d\lambda,d\theta)=k_x(\lambda,\theta)d\lambda d\theta+\nu_x^s(d\lambda,d\theta.)$) Now that we know ν_x and ν_y are a.c. with respect to Lebesgue measure simultaneously the question is how are their spectral measure related? The following proposition addresses this question.

Proposition 3.4. *If the ssorf $x_{m,n}$ has a spectral density $k_x(\lambda,\theta)$ then its transform $y=xoQ$ via Q has spectral density*
(3.2)
$$g_x^Q(\lambda,\theta):=\sum_{(j,k)\in J(Q)}\chi_{M_{j,k}}(\lambda,\theta)k_x(D\lambda-C\theta+2jD\pi-2kC\pi,A\theta-B\lambda+2kA\pi-2jB\pi).$$
This can be summarized as
$$\mu_x^Q(\lambda,\theta)=g_x^Q(\lambda,\theta)d\lambda d\theta=k_y(\lambda,\theta)d\lambda d\theta=\nu_y(\lambda,\theta).$$

Corollary 3.5. *If q is a nonpositive integer and $x_{m,n}$ is a ssorf with spectral measure ν_x then its transform $u_{m,n}=x_{m+qn,n}$ has spectral measure*

$$\mu_x^q(E):=\sum_{l=q}^{0}\nu_x((L^{-1}S_{0,l}(E\cap M_{0,l})),\ \ E\subseteq T. \tag{3.3}$$

If $x_{m,n}$ has spectral density $k_x(\lambda,\theta)$ then its transform u also has spectral density and it is given by

$$g_x^q(\lambda,\theta):=\sum_{l=q}^{0}\chi_{M_{0,l}}(\lambda,\theta)k_x(D\lambda-C\theta-2lD\pi,A\theta-B\lambda+2lB\pi);\ q\in N. \tag{3.4}$$

This may be summarized as $\nu_u(d\lambda,d\theta)=\mu_x^q(d\lambda,d\theta)$ and $k_u(\lambda,\theta)=g_x^q(\lambda,\theta)$.

Corollary 3.6. *If q is a nonepositive integer and $x_{m,n}$ is a ssorf with spectral measure ν_x then its transform $z_{m,n}=x_{m-n,n}$ has spectral measure*

$$\mu_x(E):=\nu_x(L^{-1}(E\cap T_{sw}))+\nu_x(L^{-1}((E\cap T_{ne})+(0,-2\pi))), \tag{3.5}$$

where T_{sw} and T_{ne} are the triangles in south west and north east half of T, respectively. If $x_{m,n}$ has spectral density $k_x(\lambda,\theta)$ then its transform z also has spectral density and it is given by

$$g_x(\lambda,\theta):=\chi_{T_{sw}}k_x(D\lambda-C\theta,A\theta-B\lambda)+\chi_{T_{ne}}k_x(D\lambda-C\theta-2D\pi,A\theta-B\lambda+2B\pi). \tag{3.6}$$

This may be summarized as $\nu_z(d\lambda,d\theta)=\mu_x(d\lambda,d\theta)$ and $k_z(\lambda,\theta)=g_x(\lambda,\theta)$.

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SIMULTANEOUS TRIANGULARIZATION IN THE SETTING OF BANACH AND HILBERT SPACES – A SHORT SURVEY

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Abstract. Let $\mathcal{X}$ be an arbitrary Banach space. A collection $\mathcal{F}$ of bounded operators on $\mathcal{X}$ is called *simultaneously triangularizable* or simply *triangularizable* if there exists a maximal chain of subspaces of $\mathcal{X}$ each of which is invariant under the collection $\mathcal{F}$. In this talk, I will touch upon the notion of simultaneous triangularization in both the Banach and Hilbert space settings hereby I will attempt to present a short survey of the subject.

1. Introduction and preliminaries

The notion of simultaneous triangularization in infinite dimensions has been extensively studied in the literature (see [3] for a nice exposition of the subject, and the references therein). To set the stage, let us begin by establishing some definitions and standard notation. Throughout this talk, unless otherwise stated, $\mathcal{X}$ stands for a separable real or complex Banach space. As is usual, by $\mathbb{F}$ we mean $\mathbb{R}$ or $\mathbb{C}$. The terms *subspace* and *operator* or *linear operator* will, respectively, be used to describe a closed subspace of a Banach space $\mathcal{X}$ and a bounded linear operator on $\mathcal{X}$.

We use $\mathcal{B}(\mathcal{X})$ to denote the set (in fact the algebra) of bounded operators on $\mathcal{X}$; $\mathcal{B}_0(\mathcal{X})$ is used to denote the set (in fact the ideal) of compact operators on $\mathcal{X}$, $\mathcal{B}_{00}(\mathcal{X})$ is used to denote the set (in fact the ideal) of finite-rank operators on $\mathcal{X}$.

A subspace $\mathcal{M}$ is *invariant* for a collection $\mathcal{F}$ of bounded operators if $T\mathcal{M} \subseteq \mathcal{M}$ for all $T \in \mathcal{F}$. A collection $\mathcal{F}$ of bounded operators is called *reducible* if $\mathcal{F} = \{0\}$ or

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* Invited Speaker.

it has a nontrivial invariant subspace. This definition is slightly unconventional, but it simplifies some of the statements in what follows.

In finite dimensions, it is easily seen that triangularizability of a family of linear operators is equivalent to the existence of a basis for the space such that all operators in the family have upper triangular matrix representation with respect to that basis. Note that a collection of triangularizable operators is not necessarily triangularizable. Plainly, a family $\mathcal{F}$ of linear operators is triangularizable iff $\mathrm{Sem}(\mathcal{F})$, *the semigroup generated by* $\mathcal{F}$, is triangularizable; or iff $\mathrm{Alg}(\mathcal{F})$, *the algebra generated by* $\mathcal{F}$, is triangularizable.

We also need to recall some definitions and standard facts from the theory of the complexifications of real Banach spaces. We refer the interested reader to [2] for a nice account of the theory. However, we will be dealing only with the Taylor complexification of real Banach (resp. Hilbert) spaces which will be described shortly. Let $\mathcal{X}$ be a real vector space. We use the symbol $\widetilde{\mathcal{X}}$ to denote $\mathcal{X} \times \mathcal{X}$, *the algebraic complexification of* $\mathcal{X}$, whose construction resembles that of $\mathbb{C}$ from $\mathbb{R}$, as follows

$$(x, y) + (u, v) := (x + u, y + v), (a + ib)(x, y) := (ax - by, bx + ay),$$

where $x, y, u, v, \in \mathcal{X}$ and $a, b \in \mathbb{R}$. It is easily verified that $\widetilde{\mathcal{X}}$ is a complex vector space into which $\mathcal{X}$ embeds via the mapping $x \to (x, 0)$. With that in mind, we can use the familiar notation $z = x + iy$ to denote the vector $z = (x, y)$ in $\widetilde{\mathcal{X}}$. Thus, if $z = x + iy$, it is natural to define $\mathrm{Re}(z) := x$ and $\mathrm{Im}(z) := y$. Also, in a natural way, by *the conjugate* of an element $z = x + iy$ of $\widetilde{\mathcal{X}}$, we mean the element $\bar{z}$ defined by $\bar{z} := x - iy$. A norm $||.||_{\widetilde{\mathcal{X}}}$ on $\widetilde{\mathcal{X}}$ is called a *reasonable complexification norm* provided that

$$||\mathrm{Re}(z)||_{\widetilde{\mathcal{X}}} = ||\mathrm{Re}(z)||, \ ||\bar{z}||_{\widetilde{\mathcal{X}}} = ||z||_{\widetilde{\mathcal{X}}},$$

for all $z \in \widetilde{\mathcal{X}}$, where $||.||$ denotes the norm of $\mathcal{X}$. It is not difficult to check that the norm $||.||_T$, called *the Taylor [complexification] norm* of $\widetilde{\mathcal{X}}$, defined by

$$||x + iy||_T := \sup_{a^2+b^2=1} ||ax + by||,$$

where $x, y \in \mathcal{X}$, is in fact the smallest reasonable complexification norm on $\widetilde{\mathcal{X}}$ [2, Proposition 3]. Let $\mathcal{H}$ be a real Hilbert space. It is easily seen that the norm defined by

$$||x + iy|| := (||x||^2 + ||y||^2)^{1/2},$$

is a reasonable complexification norm on $\widetilde{\mathcal{H}}$ which comes from the following inner product on $\widetilde{\mathcal{H}}$

$$\langle x + iy, u + iv\rangle := \langle x, u\rangle + i\langle y, u\rangle - i\langle x, v\rangle + \langle y, v\rangle.$$

We call $\widetilde{\mathcal{H}}$ *the natural complexification* of $\mathcal{H}$.

From this point on, unless otherwise stated, the symbol $\widetilde{\mathcal{X}}$ stands for the Taylor complexification of the real Banach space $\mathcal{X}$, and $\widetilde{\mathcal{H}}$ for the natural complexification of the real Hilbert space $\mathcal{H}$. It is easily checked that $\lim_n(x_n + iy_n) = x + iy$ in $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$) iff $\lim_n x_n = x$ and $\lim_n y_n = y$ in $\mathcal{X}$ (resp. in $\mathcal{H}$). Therefore, $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$) is a complex Banach (resp. Hilbert) space. If $T \in \mathcal{B}(\mathcal{X})$ (resp. $T \in \mathcal{B}(\mathcal{H})$), then the operator $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{X}})$ (resp. $\widetilde{T} \in \mathcal{B}(\widetilde{\mathcal{H}})$) defined by $\widetilde{T}(x + iy) := Tx + iTy$ is a bounded operator and furthermore $||\widetilde{T}|| = ||T||$ (see [2, Proposition 4]). As a matter of fact, the tilde is a covariant functor from the category of real Banach (resp. Hilbert) spaces into the category of complex Banach (resp. Hilbert) spaces.

We need the following proposition in what follows.

Proposition 1.1. *Let $\mathcal{X}$ (resp. $\mathcal{H}$) be a real Banach (resp. Hilbert) space, $T \in \mathcal{B}(\mathcal{X})$ (resp. $T \in \mathcal{B}(\mathcal{H})$), and $\widetilde{T}$ its extension to $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$). Then the following hold.*

(i) $\operatorname{rank}(T) = \operatorname{rank}(\widetilde{T})$ *and* $\dim\ker(T) = \dim\ker\widetilde{T}$.

(ii) The operator T is compact iff $\widetilde{T}$ is compact.

(iii) If $\mathcal{M}$ is a subspace of $\mathcal{X}$, then $\mathcal{M} + i\mathcal{M}$ is a subspace of $\widetilde{\mathcal{X}}$. Conversely, if $\mathcal{M}$ is a subset of $\mathcal{X}$ such that $\mathcal{M} + i\mathcal{M}$ is a subspace of $\widetilde{\mathcal{X}}$, then $\mathcal{M}$ is a subspace of $\mathcal{X}$.

(iv) A chain $\mathcal{C} = \{\mathcal{M}\}_{\mathcal{M}\in\mathcal{C}}$ is a maximal chain of subspaces of $\mathcal{X}$ iff $\widetilde{\mathcal{C}} := \{\mathcal{M} + i\mathcal{M}\}_{\mathcal{M}\in\mathcal{C}}$ is a maximal chain of subspaces for $\widetilde{\mathcal{X}}$.

(v) The bounded operator T is a trace class operator on $\mathcal{H}$ iff $\widetilde{T}$ is a trace class operator on $\widetilde{\mathcal{H}}$. Moreover, $\operatorname{tr}(T) = \operatorname{tr}(\widetilde{T})$.

(vi) A subspace $\mathcal{M}$ is invariant under T iff the subspace $\mathcal{M} + i\mathcal{M}$ is invariant under $\widetilde{T}$.

(vii) If T is compact and triangularizable, then $\sigma(T) = \sigma(\widetilde{T})$. Also, if T is compact and $\sigma(\widetilde{T}) \subset \mathbb{R}$, then T is triangularizable and $\sigma(T) = \sigma(\widetilde{T})$. Moreover, the compact triangularizable operators T and $\widetilde{T}$ share the same set of eigenvalues counting multiplicity.

We conclude this section by quoting two standard triangularization results.

Theorem 1.2. *Let $\mathcal{X}$ be a complex Banach space and $\mathcal{F}$ a family of compact operators on $\mathcal{X}$. Then, the collection $\mathcal{F}$ is triangularizable iff $P(A, B)(AB - BA)$ is quasinilpotent for all $A, B \in \mathcal{F}$ and all polynomials P in two noncommutative indeterminates with coefficients from $\mathbb{C}$.*

Theorem 1.3 (**Turovskii's Theorem**). *Let $\mathcal{X}$ be a complex Banach space. Then every multiplicative semigroup $\mathcal{S}$ of compact quasinilpotent operators on $\mathcal{X}$ is triangularizable.*

2. Main results

In the talk, we will indicate how one can use the following result to prove that every triangularizability result on certain collections of compact operators on a complex Banach (resp. Hilbert) space gives rise to its counterpart on a real Banach (resp. Hilbert) space.

Theorem 2.1. *Let $\mathcal{X}$ (resp. $\mathcal{H}$) be a real Banach (resp. Hilbert) space, $\mathcal{F}$ a family of triangularizable compact operators on $\mathcal{X}$ (resp. $\mathcal{H}$). Then $\mathcal{F}$ is triangularizable over $\mathcal{X}$ (resp. $\mathcal{H}$) iff the family $\widetilde{\mathcal{F}}$, the family consisting of the extensions of the members of $\mathcal{F}$ to $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$), is triangularizable over $\widetilde{\mathcal{X}}$ (resp. $\widetilde{\mathcal{H}}$).*

The following gives a criterion for triangularizability of an R-algebra of compact operators with spectra in R, where R is a subring of $\mathbb{R}$.

Theorem 2.2. *Let $\mathcal{X}$ be a real or complex Banach space, R a subring of $\mathbb{R}$, and $\mathcal{A}$ an R-algebra in $\mathcal{B}_0(\mathcal{X})$ with spectra in R. Then $\mathcal{A}$ is triangularizable if and only if every element of $\mathcal{A}$ is triangularizable.*

Recall that a semigroup (resp. algebra) of compact quasinilpotent operators on a Banach space is called a *Volterra semigroup* (resp. *Volterra algebra*). In the talk, we present a new proof of the following well-known lemma which is due to Radjavi and extends Kaplansky's Theorem ([3, Corollary 2.2.3]) to trace class operators

Theorem 2.3 (**Radjavi**). *Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{S}$ a semigroup in $\mathcal{C}_1$, the ideal of trace class operators, on which trace is constant. Then, the semigroup $\mathcal{S}$ is triangularizable. In particular, if trace is zero on a semigroup $\mathcal{S}$ in $\mathcal{C}_1$, then the algebra generated by $\mathcal{S}$ is a Volterra algebra of $\mathcal{C}_1$ operators.*

The following is a quick consequence of the preceding lemma.

Corollary 2.4. *Let $\mathcal{H}$ be an arbitrary Hilbert space. If an algebra $\mathcal{A}$ in $\mathcal{C}_1$ is spanned by its quasinilpotent members as a vector subspace of $\mathcal{C}_1$, then the algebra $\mathcal{A}$ is a Volterra algebra of $\mathcal{C}_1$ operators, and therefore it is triangularizable.*

Remark. By results of Fong and Sourour (see [1, Corollary 2 and Theorem 3]) every compact (resp. Hilbert-Schmidt, i.e., $\mathcal{C}_2$) operator on an infinite dimensional Hilbert space is a sum of two compact (resp. Hilbert-Schmidt) quasinilpotent operators. This would imply that the ideal of compact (resp. Hilbert-Schmidt) operators on an infinite dimensional Hilbert space, which is obviously irreducible, as a vector space, is spanned by its quasinilpotent members. Therefore, the preceding corollary cannot be generalized to algebras of compact (resp. $\mathcal{C}_p$, $p > 1$) operators on infinite dimensional Banach (resp. Hilbert) spaces.

A consequence of the preceding corollary is the following which can be thought of as a generalization of Kolchin's Theorem ([3, Theorem 2.1.8]) to $\mathcal{C}_1$ operators on a real or complex Hilbert space.

Corollary 2.5. *(i) Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{F}$ a family of $\mathcal{C}_1$ class operators on $\mathcal{H}$ with the following properties: (a) every $A \in \mathcal{F}$ has trace zero (resp. can be written as a linear combination of quasinilpotent elements from the algebra generated by $\mathcal{F}$); (b) if A and B are in $\mathcal{F}$, then $AB + A + B$ is in $\mathcal{F}$. Then $\mathcal{F}$ is triangularizable.*

(ii) Let $\mathcal{H}$ be a real or complex Hilbert space and $\mathcal{F}$ be a family of $\mathcal{C}_1$ class operators on $\mathcal{H}$ such that every A in $\mathcal{F}$ has trace zero (resp. can be written as a linear combination of quasinilpotent elements from the algebra generated by $\mathcal{F}$). Then, every semigroup of operators of the form $I + Q$ with $Q \in \mathcal{F}$ is triangularizable.

Remarks. 1. A proof identical to that of the corollary shows that the counterpart of the corollary holds for collections of finite rank operators on an arbitrary Banach space and for collections of matrices in $M_n(F)$, where F is a field whose characteristic is zero or greater than n.

2. The proof of the corollary together with Radjavi's Trace Theorem ([3, Theorem 2.2.1]) implies the following generalization of Kolchin's Theorem in finite dimensions. *Let $n \in \mathbb{N}$, F a field with $\mathrm{ch}(F) > n/2$ or $= 0$, and $\mathcal{F}$ a family of triangularizable matrices in $M_n(F)$ with trace zero. Then, every semigroup of triangularizable matrices of the form $I + A$ with $A \in \mathcal{F}$ is triangularizable.*

The following is the counterpart of Theorem 2.3.2 of [6] over arbitrary Banach spaces.

Theorem 2.6. *Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{S}$ an irreducible semigroup of compact operators on $\mathcal{X}$, and $\mathcal{I}$ a nonzero semigroup ideal of $\mathcal{S}$. Then*

$(i)\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(\mathcal{I}\mathcal{A}\mathcal{I}) = 0\} = \{0\}$.
$(ii)\{A \in \mathrm{Alg}_{\mathbb{F}}(\mathcal{S} \cup \{I\}) : \rho(A\mathcal{I}) = 0\} = \{0\}$.

The following extends Guralnick's Theorem, which is, itself, an extension of a well-known theorem of McCoy, to compact operators (resp. $\mathcal{C}_p$ class operators ($p \geq 1$)) on a real or complex Banach (resp. Hilbert) space.

Theorem 2.7. *(i) Let $\mathcal{X}$ be a real or complex Banach space, $\mathcal{C}$ a collection of compact triangularizable operators, and $m \in \mathbb{N}$. Then, $\mathcal{C}$ is triangularizable iff $(AB - BA)C$ is quasinilpotent for all $A, B \in \mathcal{C}$ and $C \in \big(\mathrm{Sem}(\mathcal{C})\big)^m$.*

(ii) Let $\mathcal{H}$ be a real or complex Hilbert space, $\mathcal{C}$ a collection of $\mathcal{C}_p$ class operators with $p \geq 1$, and $m \in \mathbb{N}$ with $m > p$. Then, $\mathcal{C}$ is triangularizable iff $\mathrm{tr}((AB - BA)C) = 0$ for all $A, B \in \mathcal{C}$ and $C \in \big(\mathrm{Sem}(\mathcal{C})\big)^m$.

We now present the following result which is a slight generalization of Radjavi's Trace Theorem (see [3, Theorem 8.6.9]).

Theorem 2.8 (**Radjavi's Trace Theorem).** *Let $\mathcal{H}$ be a real or complex Hilbert space, and $\mathcal{F}$ a family of triangularizable $\mathcal{C}_p$ operators with $p \geq 1$. Then $\mathcal{F}$ is triangularizable if and only if trace is permutable on $\mathcal{F}^m$ for some integer $m \geq p$.*

We conclude with the following result which shows that the notion of simultaneous triangularization for collections of compact operators remains intact under taking certain limit operations.

Theorem 2.9 (**Near Triangulaizability Theorem).** *Let $\mathcal{X}$ be an arbitrary Banach space, $\mathcal{F}_i, \mathcal{F}$ ($i \in \mathbb{N}$) nonempty families of compact operators on $\mathcal{X}$ such that each family $\mathcal{F}_n$ ($n \in \mathbb{N}$) is triangularizable and that $\lim_n \mathrm{dist}(\mathcal{F}_n, f) = 0$ for all $f \in \mathcal{F}$. Then, $\mathcal{F}$ is triangularizable.*

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VECTOR VARIATIONAL-LIKE INEQUALITIES AND APPLICATIONS

JAFAR ZAFARANI

ABSTRACT. Using a generalized Fan's KKM theorem, some existence results for generalized vector variational-like inequalities in noncompact settings are established. Furthermore, some applications to vector optimization problems are given. The results presented in this paper extend and unify corresponding results of other authors.

1. Introduction and preliminaries

Since Giannessi (1980) introduced the vector variational inequality, (for short,VVI) in finite dimensional Euclidian space, many authors have intensively studied (VVI) and its various extensions. Several authors have investigated relationships between (VVI) and vector optimization problems, vector complementarity problem. For details we refer to Huang and Fang(2005), and Zeng and Yao(2006) and reference therein. The vector variational-like inequalities (for short, VVLI), a generalization of (VVI) was studied by Ansari, Siddiqi and Yao(2000), and Jabarootian and Zafarani(2006). Minty's Lemma has been shown to be an important tool in the variational field including variational inequality problems, obstacle problems, confined plasmas, free boundary problems, elasticity problems and stochastic optimal control problems when the operator is monotone and the domain is convex; see Baiocchi and Capelo (1984). Lee and Lee (1999) obtained a vector version of Minty's lemma using Nadler's result(1969) and with their result they considered two kinds of vector variational-like inequalities for set-valued mappings under certain pseudomonotonicity condition and certain new hemicontinuity condition, respectively. Motivated by the above works, we first obtain two vector versions of Minty's Lemma and deduce existence theorems for the solvability of three classes of vector variational-like inequalities in normed spaces. In fact we prove the solvability results for these classes of generalized vector

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variational-like inequalities under certain pseudomonotonicity assumptions. We also prove the solvability of these classes of generalized vector variational-like inequalities without monotonicity assumptions.

Let X and Y be two normed spaces and let $L(X,Y)$ denote the family of all continuous linear operators from X into Y equipped with the uniform convergence norm. When Y is the set $\mathbb{R}$ of real numbers, $L(X,Y)$ is the usual dual space X^* of X. For any $x \in X$ and any $u \in L(X,Y)$, we shall write the value $u(x)$ as $\langle u, x\rangle$. We suppose throughout this paper that K is a nonempty and convex subset of X, $T : K \to L(X,Y)$ is a set-valued mapping, $\eta : K \times K \longrightarrow X$ and $f : K \times K \longrightarrow Y$ are functions, and $\{C(x) : x \in K\}$ is a family of closed, convex and pointed cones of Y.

Let C be a closed, convex and pointed cone with int$C \neq \emptyset$. Then a partial order $\leq_C$ in Y is defined as for $y_1, y_2 \in Y$

$$y_1 \leq_C y_2 \quad \Leftrightarrow \quad y_2 - y_1 \in C.$$

Note that $C \neq Y$ iff $0 \notin$ intC.

The purpose of this article is to prove the existence of solutions to the following three kinds of vector variational-like inequalities:

Problem (I):Find $x_0 \in K$ such that $\langle T(y), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0),\ \forall y \in K.$

Problem (II): Find $x_0 \in K$ such that $\langle T(x_0), \eta(y, x_0)\rangle + f(y, x_0) \nsubseteq -\text{int}C(x_0),\ \ \forall y \in K.$

Problem (III): Find $x_0 \in K$ such that $\langle T(y), \eta(x_0, y)\rangle + f(x_0, y) \nsubseteq \text{int}C(x_0),\ \ \forall y \in K.$

2. Main results

In this section, we prove the solvability of (VVLI) with monotone set-valued mappings.

Theorem 2.1. *Assume that $T : K \to L(X,Y)$ is η-f pseudomonotone type(I), H-hemicontinuous and compact valued. If the following conditions are satisfied:*
(i) The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus -\{intC(x)\}$ is $w \times \tau$ - closed, where w is the weak topology of X.
(ii) f and η are weak-norm continuous in the second argument
(iii) For each $x, y \in K$, $\langle T(y), \eta(x,x)\rangle + f(x,x) \subseteq -C(x)$ and $\langle T(x), \eta(x,x)\rangle + f(x,x) = \{0\}$.
(iv) For any fixed $x, y, z \in K$ the set-valued mapping $y \rightrightarrows \langle T(z), \eta(y,x)\rangle + f(y,x)$ is $C(x)$-convex.
(v) There exist a nonempty weak compact set $M \subset K$, and a nonempty weak compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that

$$\langle T(y), \eta(y,x)\rangle + f(y,x) \subseteq -intC(x).$$

Then Problem (II) holds.

Remark 2.2. Theorem 2.1 generalizes Theorem 2.1 of Huang and Fang (2005). It also improves Theorem 2. 1 of Zeng and Yao (2006) if we replace their mapping ToA, by our mapping T.

In the following we will establish another vector version of Minty's Lemma.

Lemma 2.3. *Let X and Y be two normed spaces. Assume that $T : K \rightrightarrows L(X,Y)$ is η-f pseudomonotone type(II) and H-hemicontinuous with compact values. If*
(i) For each $x, y \in K$, $\langle T(y), \eta(y,y)\rangle + f(y,y) \subseteq C(x)$.
(ii) For each $x, y, z \in K$, the set-valued mapping $y \rightrightarrows \langle T(z), \eta(y,z)\rangle + f(y,z)$ is C(x)-convex.
(iii) f and η are continuous.
Then, Problems (II) and (III) are equivalent.

Remark 2.4. If for each $x, y, z \in K$, the mapping $y \longmapsto \langle T(z), \eta(y,x)\rangle + f(y,x)$ is affine, then condition (ii) is satisfied. Hence, the above Lemma improves Theorem 3.1 of Khan et al. (2004) and therefore Theorem 2.3 of Lee and Lee (1999), if we replace their mapping ToA, by our mapping T. In the proof of Lemma 2.2 in condition (iii), the continuity of f and η in the second argument is sufficient. Lemma 2.2 is also a vector version of Lemmas 6.1 and 6.2 of Jabarootian and Zafarani (2006).

Theorem 2.5. *Let X and Y be normed spaces. Assume that all of the conditions of Lemma 2.2 are satisfied and*
(i) The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus \{intC(x)\}$ is closed.
(ii) There exist a nonempty compact set $M \subset K$, and a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there is $y \in B$ such that

$$\langle T(y), \eta(x,y)\rangle + f(x,y) \subseteq intC(x).$$

Then Problem (II) holds.

Theorem 2.6. *Let $K \subset \mathbb{R}^m$ and $g : K \to \mathbb{R}^n$ be an invex function with respect to η on K. Assume that the following conditions are satisfied:*
(i)The set-valued mapping $W : K \rightrightarrows Y$ defined by $W(x) = Y \setminus \{-intC(x)\}$ is closed.
(ii)For each $y \in K$, the function $x \longmapsto \eta(y,x)$ is continuous and η is affine in the first argument and skew function.
(iii) There is a nonempty compact set $M \subset K$, and there is a nonempty compact convex set $B \subset K$ such that for each $x \in K \setminus M$, there exists $y \in B$ such that

$$\langle \partial^c g(x_0), \eta(y,x)\rangle \subseteq -intC(x).$$

Then VOP has a generalized weakly efficient solution.

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A SHORT HISTORY OF POLISH MATHEMATICS

WIESŁAW ŻELAZKO*

Abstract. At the beginning of 20 Century Poland had almost no mathematical traditions. The only widely known Polish names were Mikołaj Kopernik (Nicolas Copernicus) - an astronomer, but during his times astronomy was treated as a branch of geometry, and Józef Hoene-Wroński - a philosopher and mathematician - whose name is connected with wronskian (Wronski determinant). In my talk I shall explain how, in spite of the above, Poland developed quite famous mathematical school and obtained quite good international position in some (chosen in advance) modern by then fields of mathematics. In particular, I shall describe creation and development of the Warsaw School of Mathematics, working mostly in topology, set theory and foundations of mathematics, and similar development of Lwów School, working mostly in functional analysis and functions of real variable. There were good chances of developping such schools also in Kraków, Wilno and Poznań, but the process was stopped by the outburst of the World War II. If time permits, I shall shortly describe the events during World War II and the process of postwar rebuilding of (very seriously damaged) Polish Mathematics

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ON SPECTRUM OF SOME ELEMENTS OF AN FLM ALGEBRA

ALI ZOHRI

Abstract. Fundamental topological spaces (also algebras) are introduced by E. Anasri-Piri in 1990 to extending the meaning of both local convexity and local boundedness. He used this new class of algebras to generalizing famous Cohen factorization theorem . In this article we will investigate about spectrum of some elements of FLM algebras. It is well known that in every Banach algebra A we have $sp_A(x) \neq \phi$ for any element $x \in A$. It is not true for more general spaces.

1. Definitions and related results

Definition 1.1. A topological linear space A is said to be fundamental if there exists $b > 1$ such that for every sequence (x_n) of A, the convergence of $b^n(x_n - x_{n-1})$ to zero in A implies that $(x_n)$is Cauchy; cf. [1, 2, 3]

Definition 1.2. A fundamental topological algebra is an algebra whose underlying topological linear space is fundamental; cf. [1, 2, 3]

Definition 1.3. A fundamental topological algebra is called locally multiplicative, if there exists a neighborhood U_0 of zero such that for every neighborhood V of zero, the sufficiently large powers of U_0 lie in V. These kind algebras are called FLM in abbreviation.

2. New results

In a Banach algebra B for every element $x \in B$, $sp(x)$ is non-void and compact [3]. Compactness of $sp(x)$ for complete metrizable FLM algebras with a unit element is proved in [2]. This is true also for m-convex Q-algebras [4, p. 60]. In this section, we would like to prove this for an FLM algebra but for some special elements.

Theorem 2.1. *In a unital FLM algebra A for any element x for which there is $b > 1$ such that $b^n x^n \to 0$ we have $sp(x) \neq \phi$.*

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