(a) A unital Banach algebra, except the algebra of complex numbers, without nontrivial idempotent.
(b) A unital Banach algebra with a nontrivial idempotent.
(Recall that 0 and 1 are called trivial idempotents.)

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We show that the Banach algebra $C(X)$ has no nontrivial idempotent iff X is connected:
Let $0 \neq f \neq 1$ be an idempotent. Then $X = f^{-1}(\{0\}) \cup f^{-1}(\{1\})$ implies that X is not connected. Conversely if X is disconnected and $X = G_1 \cup G_2$ with open disjoint sets G_1 and G_2, then $f(x) = \begin{cases} 1 & x \in G_1 \\ 0 & x \in G_2 \end{cases}$ is a trivial idempotent of $C(X)$.

Comment. If A is a (not necessarily commutative) Banach algebra with an element $a \in A$ such that $sp(a)$ is not connected, then A has a nontrivial idempotent. (cf. [B&D, Remarks of Prop. 7.9])

Ref.

[**B&D**] F.F. Bonsall, J. Duncan, complet normed algebras, Springer-Verlag,1973.