An element of a Banach algebra which has no logarithm.

Consider the unilateral shift operator $u$ on a separable Hilbert space $H$, then $u$ is Fredholm of index $\text{null } u - \text{def } u = 0 - 1 = -1$. If $\pi : B(H) \longrightarrow \frac{B(H)}{K(H)}$ is the quotient map and $\pi(u) = e^w$ for some $w$ in the Calkin algebra $\frac{B(H)}{K(H)}$, then there exists an element $w' \in B(H)$ with $\pi(w') = w$, so $\pi(u) = e^w = e^{\pi(u')} = \pi(e^{u'})$. Hence $u - e^{u'} \in K(H)$. But $e^{u'}$ is invertible and so $\text{ind } u = \text{ind } (e^{u'}) = 0$, a contradiction.