An element of a Banach algebra which has no logarithm.

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Consider the unilateral shift operator u on a separable Hilbert space H, then u is Fredholm of index $\text{nul}u - \text{def}u = 0 - 1 = -1$. If $\pi : B(H) \longrightarrow \frac{B(H)}{K(H)}$ is the quotient map and $\pi(u) = e^w$ for some w in the Calkin algebra $\frac{B(H)}{K(H)}$, then there exists an element $w' \in B(H)$ with $\pi(w') = w$, so $\pi(u) = e^w = e^{\pi(w')} = \pi(e^{w'})$. Hence $u - e^{w'} \in K(H)$. But $e^{w'}$ is invertible and so $\text{ind}u = \text{ind}(e^{w'}) = 0$, a contradiction.