An algebra can not be normed so that it becomes a Banach algebra.

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$A = C^\infty([0,1])$, the algebra of all complex valued infinitely many times continuously differentiable functions on $[0,1]$ is semisimple, for $Rad(A) = \cap_{t\in[0,1]}\{f \in C^\infty([0,1]); f(t) = 0\} = 0$. $f \mapsto f'$ is a derivation on A. The Johnson theorem says that 0 is the only derivation on a semisimple Banach algebra (cf. [$B\&D$, Theorem 18.21]). It follows that $A = C^\infty([0,1])$ is not a Banach algebra under any norm.

For a proof based on the Singer-Wermer theorem see [Sak2, Corollary 2.2.4]).

In addition a direct proof can be found in [Aup, Corollary 4.1.12].

This example is due to Šilov([sil]).

Ref.

[**Aup**] B. Aupetit,A primer on spectral theory, Springer-Verlag, 1991.

[**B&D**] F.F. Bonsall, J.Duncan, complet normed algebras, Springer-Verlag, 1973.

[**Sil**] G.E. Silov, On a property of rings of functions, DoKl. AKad. Nauk. SSSR, 58(1974),985-8.