A commutative Banach algebra A with a unique ideal; i.e. $Rad(A)$.

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Suppose that $H = L^2(0,1)$ with respect to the Lebesgue measure. Then $(Vf)(x) = \int_0^x f(t)dt$ defines an operator $V \in B(H)$ which is called Volterra operator. Clearly the closure A of $\{p(V); p$ is a polynomial in $z\}$ in $B(H)$ is the commutative Banach subalgebra of $B(H)$ generated by V and the identity operator I.

An straightforward computation shows that $(V^n f)(x) = \int_0^x \frac{(x-t)^{n-1}}{(n-1)!} f(t)dt$ and so $\| V^n \| \leq \frac{1}{(n-1)!}$ and $r(V) = \lim_n \| V^n \|^{\frac{1}{n}} = 0$. Hence $sp(B(H), V) = \{0\}$. But by [Con, VII.Theorem 5.4], sp(A,V) is equal to the polynomially convex hull of $sp(B(H), V)$, hence $sp(A, V) = \{0\}$. But the maximal ideal space of A is homeomorphic to $sp(A, V) = \{0\}$. So the only character on A is $\phi(\lambda) = \lambda$ and $\phi(x) = 0$ for $x \in A - \mathcal{C}$. Since ϕ is continuous, the unique maximal ideal space is $Rad(A) = Ker(\phi) =$ the closure of $\{p(V); p$ is a polynomial in z and $p(0) = 0\}$.

Ref.

[Con]J.B. Conway, A course in functional analysis, New York, Springer-Verlag, 1990.