A Banach algebra with a proper dense two-sided ideal.

1. $C_c(\mathcal{R}) = \{ f \in C_0(\mathcal{R}); \text{supp}(f) = \text{the closure of } \{ x \in \mathcal{R}; f(x) \neq 0 \} \text{ is compact } \}$

1. $C_0(\mathcal{R}) - C_c(\mathcal{R})$. Note that the function $f$ defined by

$$f(x) = \begin{cases} \frac{1}{1+x} & x \geq 0 \\ \frac{1}{1-x} & x < 0 \end{cases}$$

belongs to $C_0(\mathcal{R}) - C_c(\mathcal{R})$.

2. $A = \{ f \in C([0,1]); f(0) = 0 \}$ is a closed subalgebra of $C([0,1])$ not containing the constant function $1$. So $A$ is a non-unital Banach algebra. Let $f_o(t) = t$, $t \in [0,1]$. $I = \{ f_o g; g \in C[0,1] \}$ is a proper ideal of $A$ (since if

$$h(t) = \begin{cases} t \sin \frac{1}{t} & t \in (0,1] \\ 0 & t = 0 \end{cases}$$

and for some $g \in C[0,1], tg(t) = h(t)$ whenever $t \in [0,1]$ then $\lim_{t \to 0} \sin \frac{1}{t} = g(0)$, a contradiction). By the Stone-Weierstrass theorem, each $f \in A$ is the uniform limit of a sequence $(p_n)$ of polynomials with $p_n(0) = 0$. Moreover $t \rightarrow \frac{p_n(t)}{t}$

belongs to $C[0,1]$ and $t\frac{p_n(t)}{t} \rightarrow f(t)$ uniformly on $[0,1]$. So $f$ belongs to the closure of $I$. Hence $I$ is dense in $A$. 