A Banach algebra with a proper dense two-sided ideal.

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1. $C_c(\mathcal{R}) = \{f \in C_0(\mathcal{R});\ \mathrm{supp}(f) =$ the closure of $\{x \in \mathcal{R}; f(x) \neq 0\}$ is compact $\}$ is a dense ideal of $C_0(\mathcal{R})$. Note that the function f defined by

$$f(x) = \begin{cases} \frac{1}{1+x} & x \geq 0 \\ \frac{1}{1-x} & x < 0 \end{cases}$$

belongs to $C_0(\mathcal{R}) - C_c(\mathcal{R})$.

2. $A = \{f \in C([0,1]); f(0) = 0\}$ is a closed subalgebra of $C([0,1])$ not containing the constant function 1. So A is a non-unital Banach algebra. Let $f_\circ(t) = t$, $t \in [0,1]$. $I = \{f_\circ g; g \in C[0,1]\}$ is a proper ideal of A (since if

$$h(t) = \begin{cases} tsin\frac{1}{t} & t \in (0,1] \\ 0 & t = 0 \end{cases}$$

and for some $g \in C[0,1], tg(t) = h(t)$ whenever $t \in [0,1]$ then $\lim_{t \to 0} \sin\frac{1}{t} = g(0)$, a contradiction). By the Stone-Weierstrass theorem, each $f \in A$ is the uniform limit of a sequence (p_n) of polynomials with $p_n(0) = 0$. Moreover $t \longmapsto \frac{p_n(t)}{t}$ belongs to $C[0,1]$ and $t\frac{p_n(t)}{t} \longrightarrow f(t)$ uniformly on $[0,1]$. So f belongs to the closure of I. Hence I is dense in A.