A Banach algebra generated by idempotents i.e. elements x such that $x^2 = x$.

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In the following, we show that the Banach algebra $C(X)$, where X is a compact Hausdorff space, with $Card(X) > 1$,is generated by idempotents iff X is totally disconnected.

Recall that a topological space is said to be totally disconnected if for every distinct $x_1, x_2 \in X$, there exist disjoint open sets G_1 and G_2 such that $x_1 \in G_1, x_2 \in G_2$ and $X = G_1 \cup G_2$.

If X is totally disconnected, $x_1 \neq x_2, x_1 \in G_1, x_2 \in G_2, X = G_1 \cup G_2, G_1 \cap G_2 = \emptyset, G_1$ and G_2 are open, then the continuous function $f(x) = \begin{cases} 1 & x \in G_1 \\ 0 & x \in G_2 \end{cases}$ separates x_1 and x_2. So the closed self-adjoint subalgebra generated by idempotent, by the Stone-Weierstrass theorem, is $C(X)$.

Conversely, suppose that $C(X)$ is generated by its idempotents. Let x_1 and x_2 belong to X. By Urysohn's lemma there exists a function $f \in C(X)$ such that $f(x_1) = 1$ and $f(x_2) = 0$. Every element of the self-adjoint subalgebra generated by idempotents is of the form $h = \sum_{i=1}^{k} \lambda_i g_i$ (♣) for some idempotents g_i and $\lambda_i \in \mathcal{C}$. Hence there is a sequence (h_n) of elements of the form (♣) such that $h_n \longrightarrow f$ uniformly on X. So $h_n(x_1) \longrightarrow 1$ and $h_n(x_2) \longrightarrow 0$. Therefore there exists a number N such that $|h_N(x_1)| > \frac{1}{2}$ and $|h_N(x_2)| < \frac{1}{2}$. So that $x_1 \in h_N^{-1}(\{z \in \mathcal{C}; |z| > 1\}) = G_1, x_2 \in h_N^{-1}(\{z \in \mathcal{C}; |z| < 1\}) = G_2, X = G_1 \cup G_2, X = G_1 \cap G_2 = \emptyset$. Thus X is totally disconnected.