(a) A separable Banach algebra.
(b) A non-separable Banach algebra.

(a) \( \{ r_1 + ir_2; r_1, r_2 \in \mathbb{Q} \} \) is a countable dense subset of \( \mathcal{C} \). Hence \( \mathcal{C} \) is a separable Banach algebra.

(b) \( l^\infty \) isn’t separable. In fact if \( S = \{ a_1, a_2, \ldots, \} \) is a countable set in \( l^\infty \), \( a_n = (a_n^k)_{k \in \mathbb{N}} \), and \( b_n = \begin{cases} 0 & |a_n^0| \geq 1 \\ 2 & |a_n^0| < 1 \end{cases} \), then \( b = (b_n) \in l^\infty \) and for all \( n \), \( ||b - a_n||_\infty \geq |b_n - a_n^0| \geq 1 \). So that the neighborhood of \( b \) with the radius 1 doesn’t intersect \( S \). Thus \( S \) isn’t dense in \( l^\infty \).

For another proof see [A&B, Problem 25.7].

Ref.