A commutative Banach algebra whose unit ball has no extreme point (and so it isn't the dual space of any Banach space by the Krein-Milman theorem (cf. [Con, Theorem 7.4])).

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The unit ball of c_0 has no extreme point. For see this, let (x_n) belongs to the ball of c_0. $\lim_n x_n = 0$, so there exists a number N such that for all $n > N$, $|x_n| < \frac{1}{2}$. Let $y_n = z_n = x_n$ for $n \leq N$, and let $y_n = x_n + 2^{-n}$ and $z_n = x_n - 2^{-n}$ for $n > N$, then (y_n) and (z_n) belong to the unit ball of c_0 and $(x_n) = \frac{1}{2}(y_n) + \frac{1}{2}(z_n)$. So (x_n) isn't is not an extreme point.

Ref.

[**Con**]J.B. Conway, A course in functional analysis, New York, Springer-Verlag, 1990.