

Two elements x, y ($xy \neq yx$) of a Banach algebra A such that $e^x.e^y \neq e^{x+y}$.

Consider $A = B(l^2)$ and the unilateral shift operator T on l^2 , defined by $T(x_1, x_2, \dots) = (0, x_1, x_2, \dots)$ and its adjoint $T^*(x_1, x_2, \dots) = (x_2, x_3, \dots)$. Assuming $\xi_k = (\delta_{kn})_{n \in \mathcal{N}}, k \in \mathcal{N}$; $\langle e^T e^{T^*} \xi_1, \xi_1 \rangle = \langle e^T \xi_1, \xi_1 \rangle = \langle \xi_1, \xi_1 \rangle = 1$, since $T^* \xi_1 = 0$ and $T \xi_1 = \xi_2$. Also $(T + T^*)(\xi_1) = \xi_2, (T + T^*)^2(\xi_1) = \xi_1 + \xi_3, \dots$ and so $\langle e^{T+T^*} \xi_1, \xi_1 \rangle = \langle \xi_1, \xi_1 \rangle + \langle \xi_2, \xi_1 \rangle + \langle \frac{1}{2!}(\xi_1 + \xi_3), \xi_1 \rangle + \dots > 1$. Hence $e^T.e^{T^*} \neq e^{T+T^*}$.