Two elements x, y $(xy \neq yx)$ of a Banach algebra A such that $e^x.e^y \neq e^{x+y}$.

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Consider $A = B(l^2)$ and the unilateral shift operator T on l^2, defined by $T(x_1, x_2, ...) = (0, x_1, x_2, ...)$ and its adjoint $T^*(x_1, x_2, ...) = (x_2, x_3, ...)$. Assuming $\xi_k = (\delta_{kn})_{n\in\mathcal{N}}, k \in \mathcal{N}; < e^T e^{T^*}\xi_1, \xi_1 >=< e^T\xi_1, \xi_1 >=< \xi_1, \xi_1 >= 1$, since $T^*\xi_1 = 0$ and $T\xi_1 = \xi_2$. Also $(T + T^*)(\xi_1) = \xi_2, (T + T^*)^2(\xi_1) = \xi_1 + \xi_3, \ldots$ and so $< e^{T+T^*}\xi_1, \xi_1 >=< \xi_1, \xi_1 > + < \xi_2, \xi_1 > + < \frac{1}{2!}(\xi_1 + \xi_3), \xi_1 > + \ldots > 1$. Hence $e^T.e^{T^*} \neq e^{T+T^*}$.