A Banach algebra A that cannot be a (vector space) direct sum of its radical $Rad(A)$ and a Banach algebra B that is homeomorphically isomorphic with $A/Rad(A)$.

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Consider the Banach algebra l^2 and the dense subalgebra l_0^2 of l^2 consisting of the sequences which vanish out of a finite set. Let A_0 be the vector space direct sum $l_0^2 \oplus \mathcal{C}$. A_0 is an algebra with $(x,\alpha)(y,\beta) = (xy, 0), x, y \in l^2, \alpha, \beta \in \mathcal{C}$. Also $||(x,\alpha)|| = \max(||x||, |\alpha - \sum_{n=1}^{\infty} x(n)|)$ is a norm on A_0. Let A is the completion of A_0. $Rad(A) = \mathcal{C}(0,1)$. If $(x,\alpha) \in A_0$ and $[x,\alpha]$ denotes the image of (x,α) in $A/Rad(A)$, then $[x,\alpha] \mapsto x$ defines an isometric isomorphism of $A_0/RadA$ into l_0^2 which can be extended to an isometric isomorphism of $A/RadA$ onto l^2. Suppose that there exists a homeomorphic isomorphism of l^2 with a subalgebra A_1 of A. Let ξ_k denotes $\xi_k(n) = \delta_{kn}$ $(k, n \in \mathcal{N})$ and e_k denotes the corresponding element of A_1. Choose a sequence $((x_n, \alpha_n))_{n \in \mathcal{N}}$ in A_0 such that $\lim_n (x_n, \alpha_n) = e_k$ in A. Since $e_k^2 = e_k$, we have $\lim_n (x_n^2, 0) = e_k$. Thus $\lim_n x_n(k) = 0$ or 1 for all $k \in \mathcal{N}$ and also $e_k \in l_0^2$. The elements e_k are pairwise orthogonal idempotents. If $d_n = \sum_{k=1}^{n} \frac{e_k}{k}$ and $t_n = \sum_{k=1}^{n} \frac{\xi_k}{k}$, then (t_n) converges in l^2. But d_n doesn't converge in A, a contradiction.

Ref.

[Ric] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.