A non-commutative Banach algebra in which 0 is the only quasi-nilpotent.

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Let A be the free algebra on two symbols w, v, i.e. the algebra of all finite linear combinations of words in u and v. The set of all such words is countable, $\{w_n\}$, and we take the standard enumeration given by $u, v, u^2, uv, v^2, u^3, u^2v, \ldots$. Let B be the algebra of all infinite series $x = \sum_{n=1}^{\infty} \alpha_n w_n$, where $||x|| = \sum_{n=1}^{\infty} ||\alpha_n| < \infty$. Then B is a non-commutative Banach algebra. Let $x \in B, x \neq 0$, and let α_p be the first non-zero coefficient in the series $\sum_{n=1}^{\infty} \alpha_n w_n$. Then the coefficient of w_p^m in x^m is precisely α_p^m and so $||x^m|| \geq |\alpha_p|^m$ $(m = 1, 2, 3, \ldots)$, $r(x) \geq |\alpha_p| > 0$. Note that B is an infinite dimensional non-commutative Banach algebra in which the set of quasi-nilpotents coincides with the set of nilpotents.

Ref.

J. Duncan and A.W. Tullo, Finite dimensionality, nilpotents and quasi-nilpotents in Banach algebras, Proc. of the Edin. math. Soc., vol 19(Series II), Part 1, 1974.