A finite dimensional commutative algebra with nilpotent radical, an identity modulo the radical, but no global identity.

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Let $A = \mathcal{C}^2$ with multiplication $(a,b)(c,d) = (ac,0)$ $(a,b,c,d \in \mathcal{C})$. Clearly $A^2 = A$. Its radical is $R = \{(0,b); b \in \mathcal{C}\}$ and $\frac{A}{R} \simeq \mathcal{C}$. The identity of $\frac{A}{R}$ lifts to the idempotent $(1,0)$ in A [Ric, Theorem 2.3.9], but there is no identity in A.

Ref.

[**Ric**] C.E. Rickart, General theory of Banach algebras, Princeton, Van Nastrand, 1960.