A Banach algebra having no bounded approximate identity.

\[ \{xy \mid x, y \in \ell^2\} \] is a proper subset of Banach algebra \( \ell^2 \) equipped with the coordinatewise operations. In fact \( \left( \frac{1}{n} \right) \in \ell^2 \) and if \( x_n y_n = \frac{1}{n} \), then there exist an integer \( N \) such that for all \( n > N \), \( |x_n| \geq \frac{1}{\sqrt{n}} \) or for all \( n > N \), \( |y_n| \geq \frac{1}{\sqrt{n}} \), and hence \( (x_n) \not\in \ell^2 \) or \( (y_n) \not\in \ell^2 \). Now Cohen’s factorization theorem [B&D, §11. Corollary 11] implies that \( \ell^2 \) has no bounded approximate identity.

**Comment.** Using BA37, we conclude that the Banach algebra \( \ell^2 \) has neither bounded approximate identity nor unbounded one.

**Ref.**