

A semisimple commutative Banach algebra with a closed two-sided ideal I such that $\frac{A}{I}$ isn't semisimple.

Suppose that A is the algebra $C^m([0, 1])$ of all m times continuously differentiable complex-valued functions on $[0, 1]$ with the norm $\|f\| = \sum_{k=0}^m \frac{1}{k!} \sup_{x \in [0, 1]} |f^{(k)}(x)|$. Let $I = \{f \in A; f(0) = f'(0) = 0\}$. Then $\frac{A}{I}$ is not semisimple, since assuming f_0 to be $f_0(x) = x$, then $f_0^2 \in I$ and so $(f_0 + I)^2 = f_0^2 + I = 0$, hence $r(x) = \lim_n \| (f_0 + I)^n \|^{1/n} = 0$. Therefore $f_0 + I \in \text{Rad}(\frac{A}{I})$. But $f_0 + I \neq 0$. So that $\frac{A}{I}$ is not semisimple.