On a separable infinite dimensional Banach space X there exists another norm under which X isn't separable.

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Suppose that $\{e_i; i \in I\}$ is a Hamel basis for X and I is countable. For each $i \in I$, let X_i denote the linear span of $\{e_1, e_2, \cdots, e_n\}$, then $X = \cup_{i=1}^{n} X_i$. But the X_i are proper closed subspaces of X and so are nowhere dense, that is impossible by the Baire category theorem. Thus I is uncountable.
Let $a \in X, a = \sum_{i \in I} \lambda_i e_i$ where all λ except finitely many are zero. Set $\| a \|' = \sum_{i \in I} |\lambda_i|$. Then $\| \, . \, \|'$ is obviously a norm on X. For $i \neq j, \| e_i - e_j \|' = 2$ and I is uncountable, hence $(X, \| \, . \, \|')$ has no dense countable subset.