A complete metrizable linear space whose metric cannot be obtained from a norm.

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1. The linear space S consisting of all complex sequences with the metric $d((x_i),(y_i)) = \sum_{i=1}^{\infty} \frac{|x_i - y_i|}{2^i(1+|x_i-y_i|)}$ is a complete metric space. Since $d(2(1,1,\cdots),(0,0,\cdots)) \neq 2d((1,1,\cdots),(0,0,\cdots))$, the space (S,d) is not normable.

If $(X, \| \ . \ \|)$ is a normed linear space then $d(x,y) = \begin{cases} \| \ x-y \ \| +1 & x \neq y \\ 0 & x = y \end{cases}$ is a metric on X, but can not be obtained from a norm, since if x is a nonzero vector of X then $d(2x,0) \neq 2d(x,0)$.

Comment. $C^{\infty}([0,1])$ with its usual topology is a complete metrizable linear space whose topology cannot be obtained from a norm.