Two non-isometrically isomorphic spaces with the same duals. So that a such dual space could not be a W*-algebra under any multiplication and involution.

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c_0 and c are both closed subspaces of l^∞. In addition for each $x=(x_n)\in l^1, \rho_x : c_0 \longrightarrow \mathcal{C}$ given by $(y_n)\mapsto \sum_{n=1}^{\infty} x_n y_n$ is a bounded linear functional on c_0 with the norm $\| \rho_x \| = \| x \|$. Clearly $c_0^{\#}$ is isometrically isomorphic to l^1. Also for each $x=(x_n)\in l^1, \eta_x : c \longrightarrow \mathcal{C}$ given by $(y_n)\mapsto x_1 \lim_n x_n + \sum_{n=1}^{\infty} x_n y_n$ is a bounded linear functional on c with the norm $\| \eta_x \| = \| x \|$. Obviously $c^{\#}$ is isometrically isomorphic to l^1. But by BA25.DVI the closed unit ball of c_0 has no extreme point while the closed unit ball c contains at least $(1,1,1,\cdots)$ as an extreme point (since if $1 = tx_n + (1-t)y_n$ with $|x_n|\le 1$ and $|y_n|\le 1$, then $1 = tRex_n + (1-t)Rey_n$ for all n, so that $Rex_n = Rey_n = 1$ and hence $x_n = y_n = 1$ for each n). Thus c_0 and c_1 are not isometrically isomorphic.

Now by [Sak1, Corollary 1.13.3], l^1 can not be a W^*-algebra.

Re.

[Sak1] S. Sakai, C^*-algebras and W^*-algebras, Springer-Verlag, 1971.