Notation

In this site we use $X^{\#}$ for the topological dual of a normed space$X$, S' for the commutant of a subset S of $B(H)$ and T^* for the Hilbert adjoint of an operator T in $B(H)$ for any Hilbert space H.

Main Examples

(I) The set of complex numbers $\mathcal{C}$ with usual addition, multiplication and the absolute value as a norm is a unital commutative Banach algebra.

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(II) $\mathcal{C}^n$ with the coordinatewise addition, scalar multiplication and the inner product

$$< (z_1, \ldots, z_n), (w_1, \ldots, w_n) >= \sum_{i=1}^{n} z_i \overline{w_i} \tag{1}$$

is a Hilbert space.

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(III) The space $\mathcal{C}^2$ (see (II)) with the product $(a, b)(a', b') = (aa', ab' + a'b)$ is a unital commutative Banach algebra.

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(IV) Let X be a non-empty set and Y is a normed (Banach) space. Then the set $l^\infty(X, Y)$ of all bounded mappings of X into Y with the pointwise addition $(f + g)(x) = f(x) + g(x), x \in X$; poinwise scalar multiplication

$(\lambda f)(x) = \lambda f(x), \lambda \in \mathcal{C}, x \in X$; and supremum norm $\|f\| = \sup\{|f(x)|; x \in X\}$ is a normed (Banach) space. If Y is normed algebra then $l^\infty(X,Y)$ with the pointwise product $(fg)(x) = f(x)g(x)$ is a normed algebra.

We denote $l^\infty(E,\mathcal{C})$ with $l^\infty(E)$ that is a unital cammutative C^*-algebra under the involution $f^* = \overline{f}$, the conjugate of f. Also $l^\infty(\mathcal{N})$ is denoted by l^∞.

The set of all convergent sequences of complex numbers, c, is a closed *-subalgebra of l^∞ and the set of all elements of c converging to zero, c_0, is a closed *-subalgebra of c.

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(V) If X is a topological space, then the set $C_b(X)$ of all bounded continuous complex valued functions on X is a closed *-subalgebra of $l^\infty(X)$ containing the constant function 1. So $C_b(X)$ is a unital commutative C^*-algebra.

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(VI) If X is a locally compact Hausdorff space, then the set $C_0(X)$ of all continuous complex valued functions on X vanishing at infinity (i.e. for each $\varepsilon > 0$, the set $\{x \in X; |f(x)| \geq \varepsilon\}$ is compact) is a closed *-subalgebra of $l^\infty(X)$ and so is a commutative C^*-algebra.

$C_0(X)$ is unital iff X is compact. Each non-unital commutative C^*-algebra is of this form (cf. [Mur]).

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(VII) If X is a compact Hausdorff space, then the set $C(X)$ of all continuous complex functions on X is exactly $C_0(X)$ and so is a unital commutative C^*-algebra. Each unital commutative C^*-algebra is of this form (cf. [Mur]). By ([K&R1, Th. 5.3.1]), An abelian W^*-algebra is isometrically $*$-isomorphic to $C(X)$ for some extremely disconnected compact Hausdorff space X. (A topological space is called extremely disconnected or Stonean if the closure of any open set is open).

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(VIII) Let Δ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of Δ. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$ (Since if $f_n \in A(\Delta)$ and (f_n) converges to $f \in C(\Delta)$ in the norm of $C(\Delta)$ and γ is a simple closed path in the interior of Δ, then $\lim_{n\to\infty} \int_\gamma f_n(z)dz = \int_\gamma f(z)dz$ but by Cauchy's theorem $\int_\gamma f_n(z)dz = 0 (n \in \mathcal{N})$. So $\int_\gamma f(z)dz = 0$. Now Morera's theorem implies that f is analytic in the interior of Δ), and so it is a unital commutative Banach algebra. We call this the disc algebra.

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(IX) Let (Ω, μ) be a measure space and $L^p(\Omega, \mu)$ for $1 \leq p < \infty$ be the set of all complex valued measurable functions f on Ω (we assume f is equal to g if $f = g$ a.e.$[\mu]$) for which $||f||_p = (\int_\Omega |f|^p d\mu)^{\frac{1}{2}} < \infty$. $L^p(\Omega, \mu)$ with the norm $||.||_p$ is a Banach space and is a Hilbert space iff $p = 2$. $L^p(\Omega, \mu)$ denoted by $l^p(\Omega)$ if μ is counting measure. In particular, $l^p(\mathcal{N})$ denoted by l^p. Let $H = l^2$, (α_n) be a bounded sequence of complex numbers, and (ξ_n)

be the (usual) standard orthonormal basis of H, that is, $(\xi_n)(m) = \delta_{nm}$, $n, m \in \mathcal{N}$ (δ denoted the kronecker delta), so that $\zeta = \sum_{n=1}^{\infty} < \zeta, \xi_n > \xi_n$ for any $\zeta \in H$. Then the operator $T \in B(H)$ defined by $T\xi_n = \alpha_n \xi_{n+1}$ is called a weighted shift with the weights (α_n). If $\alpha_n = 1$ for all n, then T is called unilateral shift operator. It is straightforward to show that $\|T\| = \sup_n |\alpha_n|$, $r(T) = \lim_k \sup_n |\prod_{i=0}^{k-1} \alpha_{n+i}|^{1/k}$ and $T^*\xi_1 = 0$ and $T^*\xi_n = \overline{\alpha_n}\xi_{n-1}$.
If $1 \leq p < \infty$, then l^p can be regarded as a commutative Banach algebra with coordinatewise multiplication. (For $p > 1$, $\|fg\|_p \leq \|f\|_p \|g\|_p$ is a conclusion of Hőlder inequality.) The l^p, $1 \leq p < \infty$, with the involution $f \longmapsto \overline{f}$ is an involutive Banach algebra.

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(X) The Banach space $L^1([0,1])$ with the product $(fg)(x) = \int_0^x f(x-y)g(y)dy$ is a non-unital commutative Banach algebra. It is called Volterra algebra.

(XI) Let G be a locally compact group and μ a left invariant Haar measure on G, i.e. a Borel measure satisfying the following conditions.

(a) $\mu(xE) = \mu(E)$, for every $x \in E$ and every measurable $E \subseteq G$.

(b) $\mu(U) > 0$, for every non-void open set $U \subseteq G$.

(c) $\mu(K) < \infty$, for every compact set $K \subseteq G$.

With the notation IX, and under the product given by the convolution $(f * g)(s) = \int_G f(t)g(t^{-1}s)d\mu(t)$ $(s \in G)$, $L^1(G)$ is a commutative Banach algebra which called the group algebra of G. In particular, we can cansider $L^1(\mathcal{R})$, where the Lebesgue measure is an invariant Haar measure on $\mathcal{R}$. Also if G be an (algebraic) group, then G with the discrete topology is a locally compact group. A left invariant Haar measure on G is the counting measure on G. The corresponding group algebra, denoted by $l^1(G)$ and is called discrete group algebra.

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(XII) Let S be a semi-group and α a positive real-valued function on S such that $\alpha(st) \leq \alpha(s)\alpha(t) \quad (s,t \in S)$. If $l^1(S,\alpha)$ is the set of all complex-valued functions f on S for which $\sum_{s\in S}|f(s)||\alpha(s)| < \infty$, then $l^1(S,\alpha)$ with the usual pointwise addition and scalar multiplication and the product (convolution) $(f*g)(s) = \sum_{tu=s} f(t)g(u)$ (if $tu = s$ has no solutions, we assume $(f*g)(s) = 0$), and with the norm $\|f\| = \sum_{s\in S}|f(s)|\alpha(s)$is a Banach algebra. If $\alpha(s) = 1$, $l^1(S,\alpha) = l^1(S)$ is called discrete semi-group algebra, Moreover if $S = G$ is a group then $l^1(S)$ is the same discrete group algebra $l^1(G)$.

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(XIII) Let (Ω,μ) be a measure space. Then the set $L^\infty(\Omega,\mu)$ consisting of all complex valued measurable functions f on Ω (with identifying functions which are almost everywhere equal) for which $\|f\|_\infty = \inf\{\lambda; \mu\{x \in \Omega; |f(x)| > \lambda\} = 0\} < \infty$ with the essential norm $\|.\|_\infty$ and pointwise operations is a unital commutative Banach algebra.

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(XIV) If (Ω,μ) is a measure space, then $B_\infty(\Omega)$ that is the set of all bounded complex valued measurable functions on Ω is a closed subalgebra of $l^\infty(\Omega)$ and $L^\infty(\Omega,\mu)$ (again we identify almost everywhere equal functions).

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(XV) The algebra $C^m([0,1])$ of the complex valued m times continuously differentiable on $[0,1]$ with the norm $||f|| = \sum_{k=0}^{m} \frac{1}{K!} \sup_{x\in[0,1]} |f^{(k)}(x)|$ is a unital commutative Banach algebra. Its maximal ideals are precisely the $I_z = \{f; f(z) = 0\}$ where $z \in [0,1]$. Hence $C^m([0,1])$ is semi-simple.

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(XVI) Suppose W is the set of all complex-valued functions f defined on the interval $[0,2\pi]$ of the form $f(t) = \sum_{k\in\mathcal{Z}} \alpha_k \exp(ikt)$ $(t \in [0,2\pi])$, where the $\alpha_k \in \mathcal{C}$ and $\sum_k |\alpha_k| < \infty$. The set W with the usual pointwise operations and with the norm $||f|| = \sum_{k\in\mathcal{Z}} |\alpha_k|$ is a commutative Banach algebra and called the Wiener algebra. There is an isometric isomorphism between $l^1(\mathcal{Z})$ and W given by $f \longrightarrow \tilde{f}$ where $\tilde{f}(t) = \sum_{k\in\mathcal{Z}} f(k) \exp(ikt)$ $(t \in [0,2\pi])$.

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(XVII) Let X and Y are normed spaces. Then the set of all bounded linear mappings (bounded operators) from X into Y with the operator norm $||T|| = \sup\{||Tx||; ||x|| \leq 1\}$ and with the pointwise addition and scalar multiplication is a normed space. It is Banach iff Y is Banach.
If $Y = X$, the space $B(X,X) = B(X)$ with the product $(ST)x = S(Tx)$ is a normed algebra (Banach algebra, if X is a Banach space).

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(XVIII) In $(XVII)$ if $X = H$ is a Hilbert space, then $B(H)$ with the involution $T \longmapsto T^*$ being defined by $< T^*x, y >=< x, Ty >$ $(x, y \in H)$ is a C^*-algebra. Each C^*-algebra is isometrically isomorphic to a norm closed $*$-subalgebra of $B(H)$ for a Hilbert space H.

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(XIX) An operator from normed space X into normed space Y is called compact if $T(U)$ is relatively compact in Y, where U is open unit ball of X; or equivalently for each bounded sequence (x_n) in $X, (Tx_n)$ has a convergent subsequent in Y. The set of all compact operators from X into Y is denoted by $K(X,Y)$ that is a subspace of $B(X,Y)$.

If X is a Banach space, $K(X) = K(X,X)$ is a closed two-sided ideal of $B(X)$.

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(XX) Identifying $M_n(\mathcal{C})$, the algebra of all $n \times n$ matrices with entries in $\mathcal{C}$, with $B(\mathcal{C}^n) = K(\mathcal{C}^n)$. So it is a unital non-commutative $\mathcal{C}^*$-algebra.

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(XXI) Let H be a Hilbert space and $x\overline{\otimes}y$ is the (one-rank) operator given by $(x\overline{\otimes}y)z =< z,y > x$. Suppose that $(e_i)_{i\in I}$, $(f_i)_{i\in I}$ are othonormal bases for H and $(\lambda_i)_{i\in I}$ is a family of complex numbers indexed by the same set I. The operator $T = \sum_{i\in I}\lambda_i e_i\overline{\otimes}f_i$ is well-defined and belongs to $B(H)$ iff (λ_i) is bounded and then $\|T\| = \sup\{|\lambda_i|; i \in I\}$.

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(XXIa) An operator T is called of finite rank n if $n = dimT(H) < \infty$. The set $F(H)$ of all finite rank operators is a self-adjoint two-sided ideal of

$B(H)$. It is consisting of all operators as $\sum_{i\in I}\lambda_i e_i \overline{\otimes} f_i \in B(H)$ such that $\lambda_i = 0$ for all i except finitely many i.

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(XXIb) The two-sided ideal of the compact operators $K(H)$ is self-adjoint and $F(H)$ is norm-dense in $K(H)$. $K(H)$ is consisting of all operators as $T = \sum_{i\in I}\lambda_i e_i \overline{\otimes} f_i \in B(H)$ such that the λ_i are positive (the λ_i^2 are the eigenvalues of T^*T). This sum has either a finite or a denumerably infinite number of terms; in the last case, $\lambda_i \to 0$.

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(XXIc) The set $S(H)$ of all operators T for which $\sum_{i\in I}\|Te_i\|^2 < \infty$ is a self-adjoint ideal of $B(H)$. These operators are called Hilbert-Schmidt operators on H. The algebra $S(H)$ with the Hilbert-Schmidt norm $\|T\|_2 = (\sum_{i\in I}\|Te_i\|^2)^{1/2}$ is a Banach algebra. It contains operators of finite rank as a dense subset. For any pair of operators T and S in $S(H)$, the family $(< Te_i, Se_i >)_{i\in I}$ is summable. Its sum (A, B) defines an inner product in $S(H)$ and $(T, T)^{1/2} = \|T\|_2$. So $S(H)$ is a Hilbert space (independent on the choice basis (e_i)). $S(H) \subseteq K(H)$. $S(H)$ consists of precisely those compact operators $T = \sum_i \lambda_i e_i \overline{\otimes} f_i$ for which $\sum_i \lambda_i^2 < \infty$. In addition $\|T\|_2 = (\sum_i \lambda_i^2)^{1/2}$.

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(XXId) The set of all products of two Hilbert-Schmidt operators is denoted by $N(H)$ and its elements are called trace-class operators. This set is a self-adjoint two-sided ideal of $B(H)$ and coincides with the set of those operators T for which $\sum_{i\in I} < |T|e_i, e_i >< \infty$ where $|T|$ is the absolute value of T in the C^*-algebra $B(H)$. If $\|T\|_1 = \sum_{i\in I} < |T|e_i, e_i >$, then $N(H)$ with this norm is a Banach algebra. $F(H)$ is a dense subset of $N(H)$. $N(H)$ is contained in $K(H)$ and contains $S(H)$. The elements of $N(H)$ are precisely the compact operators $T = \sum_{i\in I} \lambda_i e_i \overline{\otimes} f_i$ for which $\sum_i \lambda_i < \infty$. Moreover, $\|T\|_1 = \sum_i \lambda_i$.

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(XXII) The set $\mathcal{C}[z]$ of all polynomials in an indeterminate z with complex coefficients under usual operations on polynomials and with the norm $\|p\| = \sup_{|\lambda|\leq 1} |p(\lambda)|$ is a normed algebra.

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(XXIII) The set of all formal polynomials of degree at most n with the usual addition, scalar multiplication and product (but together with the convention that $x^k = 0$ if $k > n$) and with the norm $\|p\| = \sum_{k=1}^{n} |\alpha_k|$ $(p(x) = \sum_{k=1}^{n} \alpha_k x^k)$ is a finite dimensional Banach algebra.

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(XXIV) The algebra $C([0,1])$ with the supremum norm $\|.\|$ and multiplication $(f * g)(t) = \int_0^t f(s)g(t-s)ds$ is a Banach algebra.