A non-separable von Neumann algebra with a (unique) separable closed $*$-bi-ideal.

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Let H be a separable infinite dimensional Hilbert space and (x_n) be a dense sequence in H. Then $K(H)$ which is the closed linear span of rank-one projections, is the closure of the linear span of $x_n \overline{\otimes} x_n$ with rational coefficients,hence it is separable. If (e_n) is an orthonormal basis for H and for each subset S of the natural numbers $\mathcal{N}$,

$$P_S(e_n) = \begin{cases} e_n & n \in S \\ 0 & \text{otherwise} \end{cases},$$

then $||P_S - P_{S'}|| = 1$, for $S \neq S'$. Thus $\{P_S\}_{S \in 2^N}$ cannot be in the closure of any countable sequence of $B(H)$. Thus $B(H)$ isn't separable.

Note that for x and y in H the rank one operator $x \overline{\otimes} y$ is defined by

$$(x \overline{\otimes} y)(z) =< z, y > x.$$