**An involutive Banach algebra $A$ which isn't a $C^*$-algebra.**

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Consider $A = A(\Delta)$[1]. Then $f^*(z) = \overline{f(\bar{z})}$ gives an involution on $A$ such that $\|f\| = \sup_{z\in D} |f(z)| = \sup_{z\in D} |f(\bar{z})| = \|f^*\|$. Consider $f(z) = z^2$ and $g(z) = z$, then $g$ is self-adjoint and $f = gg^*$. So $f$ is positive and we must have $sp(f) \subseteq [0, \infty)$ contradicting $sp(f) = \Delta$. Hence $A$ isn't a $C^*$-algebra.

[1]Let $\Delta$ denote the closed unit disc $\{z \in \mathcal{C}, |z| \leq 1\}$. Suppose that $A(\Delta)$ denoted the set of all elements of $C(\Delta)$ which are analytic on the interior of $\Delta$. $A(\Delta)$ is a closed subalgebra of $C(\Delta)$.