An involution # on Banach algebra $M_4(\mathcal{C})$, two normal matrix T and S such that $TS = ST$ but $TS^{\#} \neq S^{\#}T$, $S+T$ isn't normal and $\|SS^{\#}\| \neq \|S\|^2$.

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Set

$$U = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \qquad S = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad T = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Then $Q^{\#} = U^{-1}Q^*U$ where Q^* denote the conjugate transpose of Q is an involution on $M_4(\mathcal{C})$. An straightforward computation shows that S and T has desired properties.

Ref.

[**Rud1**] W. Rudin, Functional analisis, McGraw-Hill, 1989.