A closed left ideal of a C^*-algebra without any left approximate identity.

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If ξ is a unit vector in a Hilbert space H with dimension at least 2, then $\Delta = \{T \in B(H); T\xi = 0\}$ is a closed left ideal in the C^*-algebra $B(H)$. If Δ has a left approximate identity $\{S_\alpha\}$ and $\eta \neq 0$ is a vector in H such that $< \xi, \eta >= 0$, then $\xi\overline{\otimes}\eta \in \Delta$ and so $\lim_\alpha S_\alpha(\xi\overline{\otimes}\eta) = \xi\overline{\otimes}\eta$. Thus $\lim_\alpha \|(S_\alpha\xi - \xi)\overline{\otimes}\eta\| = \lim_\alpha \|S_\alpha\xi - \xi\|\|\eta\| = 0$, hence $0 = \lim_\alpha \|S_\alpha\xi - \xi\| = \|\xi\|$, a contradiction. Thus Δ has no left approximate identity.

Note that for ζ_1 and ζ_2 in H the rank one operator $\zeta_1\overline{\otimes}\zeta_2$ is defined by

$$(\zeta_1\overline{\otimes}\zeta_2)(\zeta_3) =< \zeta_3, \zeta_2 > \zeta_1.$$