Given a compact subset K of $\mathcal{C}$, there exists a bounded operator T on a Hilbert space such that $sp(T) = K$ and the set of eigenvalues of T is dense in K.

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Suppose that $H = l^2$, (e_n) is the standard orthonormal basis for H and (λ_n) is a dense sequence in K. Set $T(\sum_{n=1}^{\infty}\alpha_n e_n) = \sum_{n=1}^{\infty}\lambda_n\alpha_n e_n$ where $(\alpha_n) \in l^2$. Obviously $K \subseteq sp(T)$. If $\lambda \notin K$, then $\inf\{|\lambda - \mu|; \mu \in K\} > 0$ and so $S(\sum_{n=1}^{\infty}\alpha_n e_n) = \sum_{n=1}^{\infty}(\lambda - \lambda_n)^{-1}\alpha_n e_n$ is a well-defined operator on H. S is the inverse of $\lambda I - T$. Therefore $\lambda \notin sp(T)$. Thus $K = sp(T)$.

For every n, $Te_n = \lambda_n e_n$. In fact $\{\lambda_1, \lambda_2, \cdots\}$ is the set of all eigenvalues of T that is dense in $sp(T)$.