Operators of arbitrary large norms that are bounded by 1 on a given basis of a separable infinite dimensional Hilbert space $H$.

Let $(\xi_n)$ be an orthonormal basis for $H$. For $k \in \mathcal{N}$, define $T_k$ on $H$ by $T_k \eta = < \eta, \xi_1 + \xi_2 + \cdots + \xi_k > \xi_1$. Then

$$T_k \xi_n = \begin{cases} 
 \xi_1 & n \leq k \\
 0 & n > k 
\end{cases}$$

Hence $\| T_k \xi_n \| \leq 1 (n \in \mathcal{N})$. On the other hand $T_k^* \eta = < \eta, \xi_1 > (\xi_1 + \cdots + \xi_k)$ $(\eta \in H)$; therefore $\| T_k \| = \| T_k^* \| \geq \| T_k^* \xi_1 \| = \| \xi_1 + \cdots + \xi_k \| = \sqrt{k}$. 

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