Given a compact subset $K$ of $\mathcal{C}$ such that $\overline{K^0} = K$, there exists an operator $T$ acting on a Hilbert space $H$ such that $sp(T) = K$ and $T$ has no eigenvalue.

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Let $H = L^2(K)$ in which $K$ is equipped with the Lebesgue measure $m$ on $\mathcal{R}^2$. Define $T$ on $H$ as the following:

$$(Tf)(\mu) = \mu f(\mu); \; \mu \in K, f \in H.$$ If $\lambda \notin K$, then $\sup\{|\lambda - \mu|^{-1}; \mu \in K\} < \infty$ and so we can define an operator $S$ on $H$ by $(Sf)(\mu) = (\lambda - \mu)^{-1} f(\mu); f \in H, \mu \in K$. Hence $S(T - \lambda I) = (T - \lambda I)S = I$ so that $\lambda \notin sp(T)$. If $\lambda \in K$, $(\lambda I - T)^{-1} \in B(H)$ and $f$ denotes the characteristic function of $\{\mu; |\lambda - \mu| < \epsilon\}$ multiplied by $m(\{\mu; |\lambda - \mu| < \epsilon\})^{-1/2}$, then

$$1 = \| f \|_2 \leq \| (\lambda I - T)^{-1} \|| \| (\lambda I - T) f \|_2$$

$$= \| (\lambda I - T)^{-1} \| \int_K (\lambda - \mu) f(\mu) dm(\mu) \leq \| (\lambda I - T)^{-1} \| \epsilon,$$

a contradiction. Hence $(\lambda I - T)$ is not invertible. So $\lambda \in sp(T)$. It follows that $sp(T) = K$. In addition, if $Tf = \alpha f$ for some $\alpha \in \mathcal{C}$, then for all $\mu \in K$, $\mu f(\mu) = \alpha f(\mu)$. So $f = 0$ almost everywhere. Thus $T$ has no eigenvalue.