Given a compact subset K of $\mathcal{C}$ such that $\overline{K^0} = K$, there exists an operator T acting on a Hilbert space H such that $sp(T) = K$ and T has no eigenvalue.

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Let $H = L^2(K)$ in which K is equipped with the Lebesgue measure m on $\mathcal{R}^2$. Define T on H as the following:

$$(Tf)(\mu) = \mu f(\mu);\ \ \mu \in K, f \in H.$$

If $\lambda \notin K$, then $\sup\{|\lambda - \mu|^{-1}; \mu \in K\} < \infty$ and so we can define an operator S on H by $(Sf)(\mu) = (\lambda - \mu)^{-1} f(\mu); f \in H, \mu \in K$. Hence $S(T - \lambda I) = (T - \lambda I)S = I$ so that $\lambda \notin sp(T)$. If $\lambda \in K, (\lambda I - T)^{-1} \in B(H)$ and f denotes the characteristic function of $\{\mu; |\lambda - \mu| < \epsilon\}$ multiplied by $m(\{\mu; |\lambda - \mu| < \epsilon\})^{-1/2}$, then

$$\begin{aligned} 1 &= \| f \|_2 \leq \| (\lambda I - T)^{-1} \| \| (\lambda I - T) f \|_2 \\ &= \| (\lambda I - T)^{-1} \| \int_K (\lambda - \mu) f(\mu) dm(\mu) \leq \| (\lambda I - T)^{-1} \| \epsilon, \end{aligned}$$

a contradiction. Hence $(\lambda I - T)$ is not invertible. So $\lambda \in sp(T)$. It follows that $sp(T) = K$. In addition, if $Tf = \alpha f$ for some $\alpha \in \mathcal{C}$, then for all $\mu \in K$, $\mu f(\mu) = \alpha f(\mu)$. So $f = 0$ almost every where. Thus T has no eigenvalue.